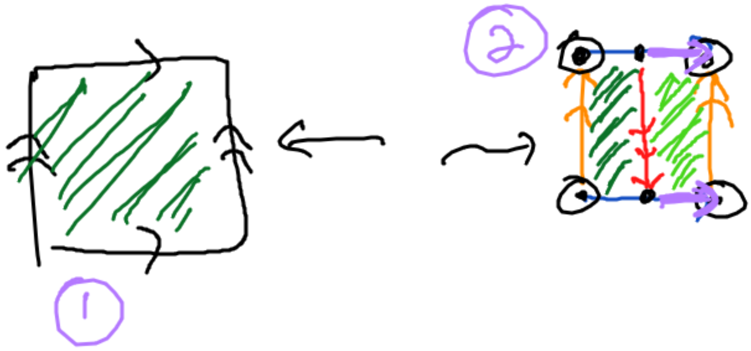


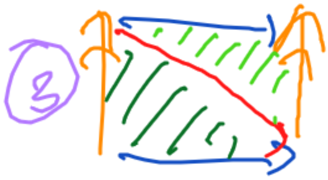
3

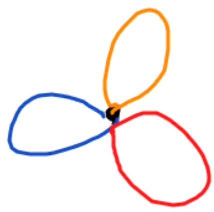
2 0-cells

4 1-cells

2 2-cells

$\boxed{0}$





1 0-cell

3 1-cells

2 2-cells.

$\boxed{0}$

Homology:

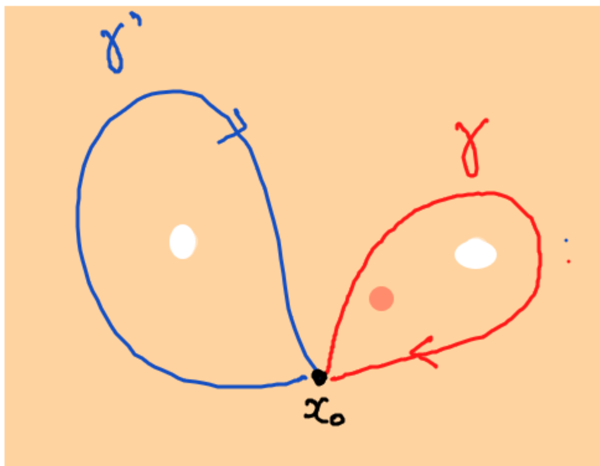
Preview: Another invariant of top spaces. (If $X \cong Y$ then their homologies will be equivalent.)

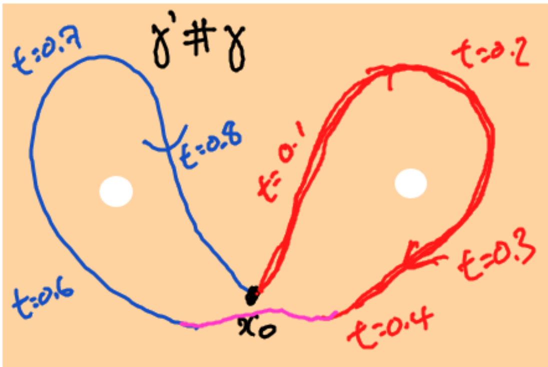
Review:

Notation	Name	Type
$\chi(X)$	Euler characteristic of X	Number (integer).
$\pi_1(X, x_0)$	Fundamental group of X	Group (Not necessarily abelian b/c $[\gamma' \# \gamma]$ may not equal to $[\gamma \# \gamma']$.)
$H_n(X)$	n^{th} homology gp of X	Abelian groups
$H_n(X; \mathbb{Z}/2\mathbb{Z})$	n^{th} homology gp of X, mod 2.	Abelian groups

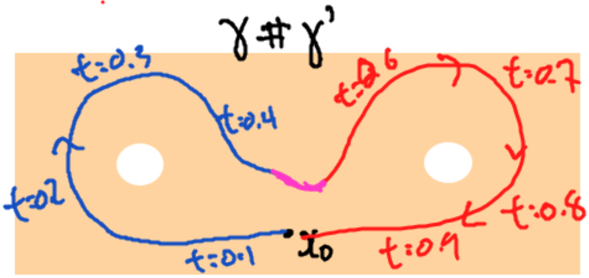
$H_0(X), H_1(X), \ldots$ are all invariants

$X = \mathbb{R}^2 \setminus \{(0,0), (5,0)\}$





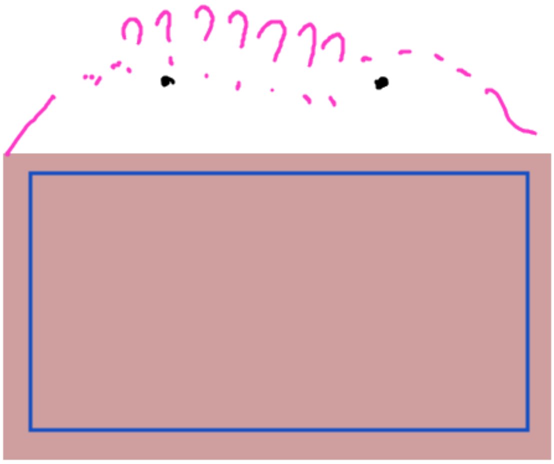
$\mathbb{Z} * \mathbb{Z}$



$ghg^{-1}h^{-1} = e$

$\Rightarrow ghg^{-1} = h$

$\Rightarrow gh = hg$



$\underline{ghg^{-1}h^{-1}} \neq e$

π_1 NOT abelian

$\mathbb{Z} * \mathbb{Z}$

$\mathbb{R}^2 \setminus \{p, q\}$

$f \updownarrow$

$\mathbb{R}^2 \setminus pt$

$f : \pi_1(\mathbb{R}^2 \setminus \{p,q\}) \longrightarrow \pi_1(\mathbb{R}^2 \setminus \{\cdot\})$

$ghg^{-1}h^{-1} \longmapsto f(g)f(h)f(g^{-1})f(h^{-1})$

$\qquad\qquad\qquad\qquad \shortparallel$

$e = f(g)f(g)^{-1}f(h)f(h)^{-1} = \underline{f(g)f(g^{-1})}f(h)f(h^{-1})$

Homology for CW complexes:

Fix a CW complex X. ($\cdot\, A_0, A_1, A_2, A_3, \ldots$) sets of cells

$\cdot\ \varphi_a : \partial D^n \longrightarrow X_{n-1}$ for all $a \in A_n$.

Define $\cdot$ groups $C_0, C_1, C_2, \cdots$

as follows: $C_n := \underbrace{\mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z}}_{\# A_n}$.

$$= (\mathbb{Z}/2\mathbb{Z})^{\oplus A_n} = \bigoplus_{A_n} \mathbb{Z}/2\mathbb{Z}$$

Fix two abelian groups A and B. The DIRECT SUM OF A AND B.

<u>Defn</u> $A \oplus B = A \times B = \{(a,b) \mid a \in A,\ b \in B\}$

This is an abelian gp as follows:

$$(a,b) + (a',b') := (a+a', b+b').$$

<u>Rmk</u> $(0,0)$ is the unit of $A \oplus B$

$(-a,-b)$ is the inverse of (a,b).

Ex $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ has 4 elmnts:

$$(0,0) \qquad (0,1)$$

$$(1,0) \qquad (1,1)$$

$\mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z} \ni (a_1, \ldots, a_n)$

$(a,b)+(a',b') = (a+a', b+b')$

$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

HOLLY

CARLOS

ELIZ.

ABIGAIL

PIERCE

4/B

	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

If we have an <u>infinite</u> cllxn of abelian gps

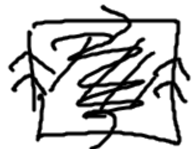
$\prod_{i\in I} A_i := \{(a_i)_{i\in I} \mid a_i \in A_i\}$ $\{A_i\}_{i\in I}$ ← infinite set

$$\bigoplus_{i\in I} A_i := \left\{ (a_i)_{i\in I} \;\middle|\; \begin{array}{l} a_i \in A_i \ \forall i \in I \\ \text{AND only finitely} \\ \text{many } a_i \text{ are NOT } 0 \end{array} \right\}$$

$$\bigoplus_{i\in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = \left\{ (\ldots, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) \;\middle|\; \begin{array}{l} a_i = 0 \text{ or } 1 \\ \text{AND} \\ \text{only finitely many } a_i \\ \text{equal } 1 \end{array} \right\}$$

$i=-3$

$$C_n := \bigoplus_{A_n} \mathbb{Z}/2\mathbb{Z}$$

Ex $X =$ 

CLAIM

Using $\{\Phi_a\}_{a \in A_n}$ we can define group homomorphism, $\partial_n : C_n \to C_{n-1}$

$\bigoplus_{A_{n-1}} \mathbb{Z}/2\mathbb{Z}$

$C_0 = \mathbb{Z}/2\mathbb{Z}$ $\quad A_0 = *$

$C_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ $\quad A_1 = \{e, f\}$

$C_2 = \mathbb{Z}/2\mathbb{Z}$ $\quad A_2 = \{T\}$

$C_3 = 0$ $\quad A_3 = \emptyset$

$(1,1,1,0,0)$ $C_4 = 0$ $\quad A_4 = \emptyset$

$\vdots$ $\quad \vdots$

CLAIM 2

$\partial_{n-1} \circ \partial_n = 0$

Idea: $C_n = \bigoplus_{A_n} \mathbb{Z}/2\mathbb{Z} \ni (0,0,\dots,0,1,0,\dots,0) \leftarrow a \in A_2$

" $(7,1,3,-2,0)$ "

$(\#,\#,\dots,\#)$

∂D^2

$a \in A_2$

$\xrightarrow{\Phi_a}$

X_1

COUNT # of times $(n-1)$ dim cells are surjected onto.

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \cdots$$

Because $\partial_{n-1} \circ \partial_n = 0$,

we know

$$\mathrm{Ker}(\partial_{n-1}) \supset \mathrm{Im}(\partial_n)$$

<u>DEFN</u> Let X be a CW cplx.

$$H_n(X; \mathbb{Z}/2\mathbb{Z}) := \mathrm{Ker}(\partial_n) / \mathrm{Im}(\partial_{n+1})$$

~~Ex~~ <u>Defn</u> The dimension of $\boxed{\underbrace{\bigoplus}_{A} \mathbb{Z}/2\mathbb{Z}}$ is $\# A$.

By Homework,

$$\sum_{n=0}^{\infty} (-1)^n \dim H_n(X; \mathbb{Z}/2\mathbb{Z}) = \overbrace{\sum_{n=0}^{\infty} (-1)^n \dim \operatorname{Ker}(\partial_n)/\operatorname{Im}(\partial_{n+1})}$$

HW

$$= \sum_{n=0}^{\infty} (-1)^n \dim C_n$$

$$= \sum_{n=0}^{\infty} (-1)^n \# A_n = \chi(X)$$