$H_*(X; \mathbb{Z}/2\mathbb{Z})$ for CW complexes.

$X = X_0 \cup X_1 \cup X_2 \cup \cdots$

Then $X_n := X_{n-1} \cup \left(\bigcup_a D^n\right)$

$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \cdots$

glue D^n to X_{n-1} along φ_a.

Fix • sets $A_0, A_1, A_2, \cdots$

$A_n :=$ the set of n-dim. cells

• $\forall\ a \in A_n$, a map $\underset{\substack{\parallel \\ S^{n-1}}}{\partial D^n} \xrightarrow{\varphi_a} X_{n-1}$

$\rightsquigarrow$ can make abelian gp maps $(\mathbb{Z}/2\mathbb{Z})^{\oplus A_n} \xrightarrow{\partial_n} (\mathbb{Z}/2\mathbb{Z})^{\oplus A_{n-1}}$

$$H_n(X; \mathbb{Z}/2\mathbb{Z}) := \mathrm{Ker}(\partial_n) / \mathrm{Image}(\partial_{n+1}).$$

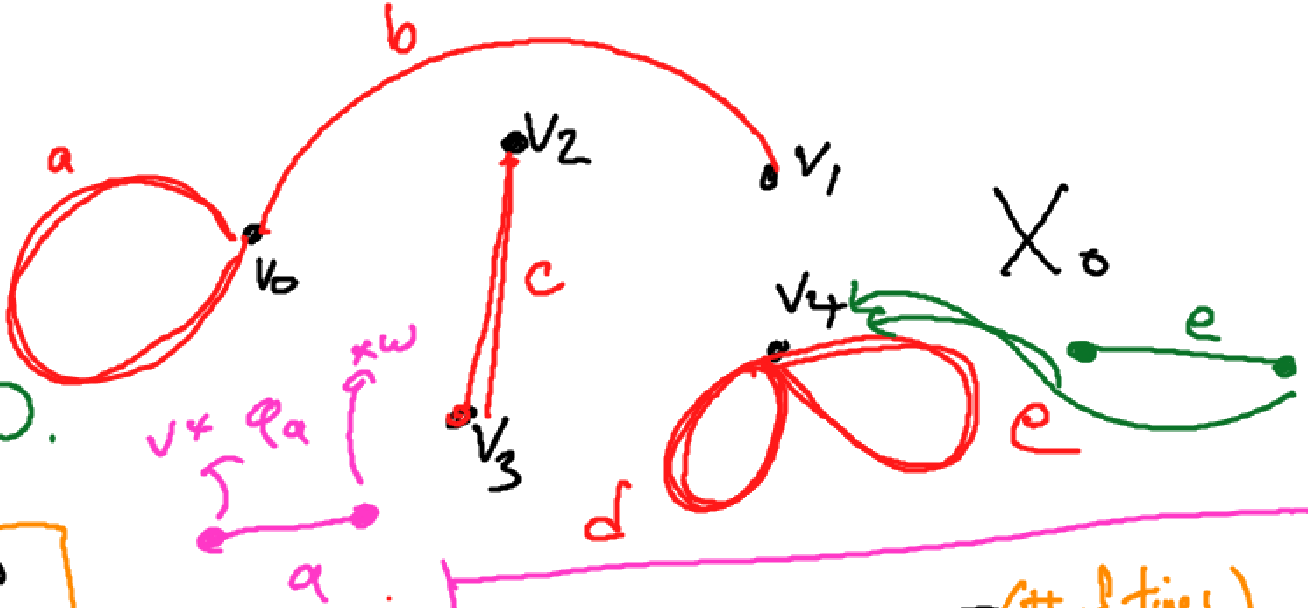
$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$C_n = (\mathbb{Z}/2\mathbb{Z})^{\oplus A_n} \qquad C_{n-1} = (\mathbb{Z}/2\mathbb{Z})^{\oplus A_{n-1}}$$

<u>Last time</u>: $H_n(X; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus A_0} & n = 0 \\ 0 & n > 0. \end{cases}$

If $X = X_0$

Suppoce $X = X_1$.



$$\partial_1(e) = 0 1_{v_0} + 0 1_{v_1} + 0 1_{v_2} + 0 1_{v_3} + \underbrace{2 1_{v_4}}_{0 \text{ in } \mathbb{Z}/2\mathbb{Z}} = 0.$$

$$\begin{array}{ccc} C_1 & \xrightarrow{\partial_1} & \boxed{C_0} \\ \| & & \| \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus A_1} & & (\mathbb{Z}/2\mathbb{Z})^{\oplus A_0} \ni 1_v \end{array}$$

$v \in A_0$

$1_a \in (\mathbb{Z}/2\mathbb{Z})^{\oplus A_1}$

$\cup$

a^{th} entry

$(0, \cdots, 0, 1, 0, \cdots, 0) =: 1_a$, $a \in A_1$.

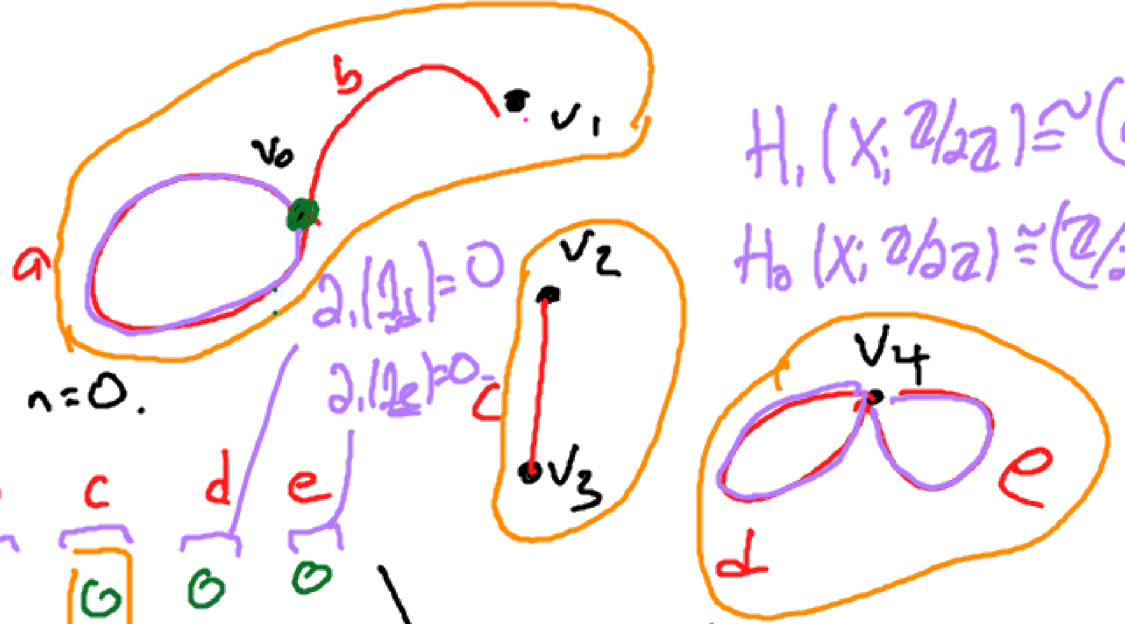
$$\boxed{\partial_1(1_a) := \sum_{v \in A_0} \left(\begin{array}{c} \# \text{ of times} \\ \varphi_a \text{ hits} \\ v \end{array} \right) 1_v.}$$

Ex

$$\partial_1(1_b) = 1 \cdot 1_{v_0} + 1 \cdot 1_{v_1} + 0 v_2 + 0 v_3 + 0 v_4$$
$$= 1_{v_0} + 1_{v_1}.$$

For $X = X_1$,

$$H_n(X;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} 0 & n \geq 2 \\ \ker(\partial_1) & n = 1 \\ C_0/\mathrm{image}(\partial_1) & n = 0. \end{cases}$$

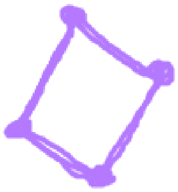


$\partial_1(1_b) = 0$

$\partial_1(1_e) = 0$

$H_1(X;\mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$

$H_0(X;\mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$



$H_1 \cong \mathbb{Z}/2\mathbb{Z}$

$$\partial_1 = \begin{array}{c|c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$\dim(\ker(\partial_1)) = 3$

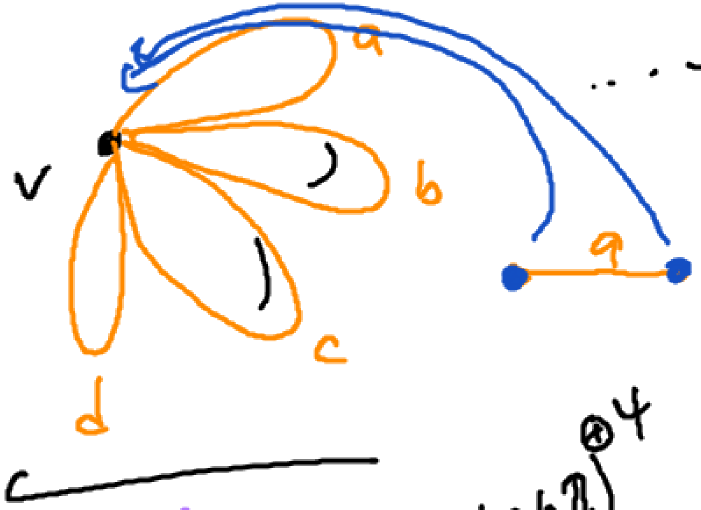
$\dim(\mathrm{Im}(\partial_1)) = 2.$

$\partial_1(1_a) = 0.$

$$\cdots \underbrace{C_3}_{=0} \to \underbrace{C_2}_{=0} \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} \underbrace{C_0}_{=\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\partial_0} 0 \to \cdots$$

$\oplus A_0$

value of $\partial_1(c)$.



X

$$\cdots \longrightarrow C_3 \xrightarrow{0} C_2 \xrightarrow{0} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{0} 0 \longrightarrow 0$$

$$C_3 = 0, \quad C_2 = 0, \quad C_1 = (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}, \quad C_0 = \mathbb{Z}/2\mathbb{Z}$$

$\mathbb{1}_a \quad \mathbb{1}_b \quad \mathbb{1}_c \quad \mathbb{1}_d$

$$\partial_1 = \begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \mathbb{1}_v \; v & \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$H_1(X;\mathbb{Z}/2\mathbb{Z}) \cong C_1 \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$

$H_0(X;\mathbb{Z}/2\mathbb{Z}) \cong C_0 \cong \mathbb{Z}/2\mathbb{Z}.$

$\partial_1(\mathbb{1}_a) = (\#$ of times ∂a hits $v) \cdot \mathbb{1}_v = 0 \cdot \mathbb{1}_v = 0.$

2

Ex $X = S^n$

$$H_k = \mathrm{Ker}(\partial_k) / \mathrm{Im}(\partial_{k+1}).$$

$n=1$

$$H_0(S^1; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

$$H_1(S^1; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

$$\forall n \geq 2 \quad H_n(\qquad) = 0$$

$$C_n := \underbrace{\mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z}}_{\# A_n}$$

$n=2$:

$A_0 = \{v\}$

$A_1 = \emptyset$

$A_2 = \{a\}$

$A_3 = A_4 = \cdots = \emptyset$

$$\begin{tikzcd}
C_3 \arrow[r, "\partial_3"] \arrow[d, equal] & C_2 \arrow[r, "\partial_2"] \arrow[d, equal] & C_1 \arrow[r, "\partial_1"] \arrow[d, equal] & C_0 \arrow[r, "\partial_0"] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\partial_2"] & 0 \arrow[r, "\partial_1"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\partial_0"] & 0
\end{tikzcd}$$

Abigail: $H_0(S^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$

Braden: $H_1(S^2; \mathbb{Z}/2\mathbb{Z}) \cong 0$

Holly: $H_2(S^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$

$$\boxed{H_3 = H_4 \cdots = 0.}$$

S^3

$\boxed{A_0 = \{v\}}$

$A_1 = \emptyset$

$A_2 = \emptyset$

$\boxed{A_3 = \{a\}}$

$\emptyset = A_4 = A_5 = A_6 = \cdots$

$$\begin{array}{cccccccccc}
C_n & C_4 & \xrightarrow{\partial_4} & C_3 & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 \\
\shortparallel & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\
\mathbb{Z}/2\mathbb{Z} & 0 & \xrightarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \mathbb{Z}/2\mathbb{Z}
\end{array}$$

CARLOS PIERCE BRANNON ELIZ ABIG.

$H_4 \cong \mathrm{Ker}(\partial_4) / \mathrm{Im}(\partial_5) = 0/0 \cong 0.$

$H_0 = \mathrm{Ker}(\partial_0) / \mathrm{Im}(\partial_1) = \mathbb{Z}/2\mathbb{Z} / 0 \cong \mathbb{Z}/2\mathbb{Z}.$

$H_1 = \mathrm{Ker}(\partial_1) / \mathrm{Im}(\partial_2) \cong 0/0 \cong 0.$

$H_2 = \mathrm{Ker}(\partial_2) / \mathrm{Im}(\partial_3) \cong 0/0 \cong 0.$

$H_3 = \mathrm{Ker}(\partial_3) / \mathrm{Im}(\partial_4) \cong \mathbb{Z}/2\mathbb{Z} / 0 \cong \mathbb{Z}/2\mathbb{Z}$

Conclusion:

$$H_k(S^n; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & k=0 \\ \mathbb{Z}/2\mathbb{Z} & k=n \\ 0 & \text{otherwise} \end{cases}$$

Corollary If $m \neq n$, then $S^m \not\cong S^n$.

$H_m \cong \mathbb{Z}/2\mathbb{Z}$ NOT homeom. $H_m = 0$.

Ex Let $n \geq 2$, and build a CW cplx as follows:

$A_0 = \{v\}$

$A_1 = \emptyset$

$\vdots$

$A_{n-1} = \emptyset$

$A_n = \{a_1, \dots, a_\ell\}$

$\emptyset = A_{n+1} = A_{n+2} = \cdots$

A bouquet of (n-dimensional) spheres.

$$H_k(X; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & k=0 \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus \ell} & k=n \\ 0 & \text{otherwise.} \end{cases}$$

$X =$ 

$$\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots \qquad C_1 \to C_0 \to 0$$

$$0 \xrightarrow{0} 0 \xrightarrow{0} (\mathbb{Z}/2\mathbb{Z})^{\oplus \ell} \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} 0$$

Rmk $H_*(X;\mathbb{Z}/2\mathbb{Z})$ is "homology mod 2"

or "homology w/ $\mathbb{Z}/2\mathbb{Z}$ coefficients."

$\boxed{2=0}$

⚠ Professionals often deal w/ $H_*(X) = H_*(X;\mathbb{Z})$

"homology" or "homology w/ $\mathbb{Z}$ coeffs."

∂,

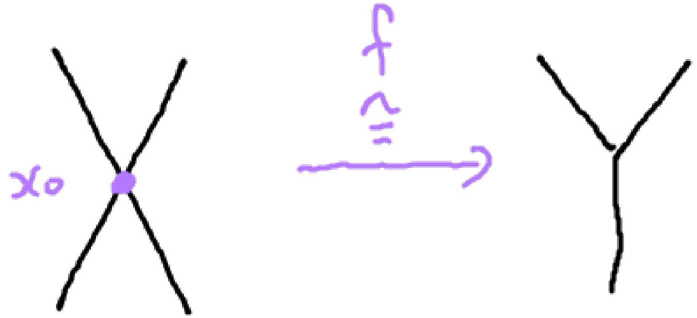
"$1+1$"

"$1-1$" $=0$

Q P

X Y R R D

O



If f is a homeo,

$$X \setminus \{x_0\} \xrightarrow{f'} Y \setminus \{f(x_0)\}$$

$$x \longmapsto f(x)$$

is a homeom.