Writing Assignment 7

Due Monday, April 5, 11:59 PM

This is a linear algebra exercise.

(a) Suppose you have three finite-dimensional vector spaces A, B, C over the same field. (You may pretend A, B, C are real vector spaces if you would like.) Suppose also that you are given two linear functions

$$f : A \to B, \qquad g : B \to C$$

where f is an injection and g is a surjection, and that $g \circ f = 0$.

Show that the dimension of the quotient vector space $\ker(g)/\operatorname{im}(f)$ is equal to the number $(-\dim A + \dim B - \dim C)$.

(b) More generally, suppose you have a collection of finite-dimensional vector spaces $V_0, V_1, V_2, \ldots, V_n$ along with linear maps $f_i : V_i \to V_{i+1}$ satisfying the conditions

(a) $f_{i+1} \circ f_i = 0$,

(b) f_0 is an injection, and

(c) f_{n-1} is a surjection.

Prove the equality

$$\sum_{i=0}^{n} (-1)^i \dim(\ker(f_i)/\operatorname{im}(f_{i-1})) = \sum_{i=0}^{n} (-1)^i \dim V_i.$$

(For the purposes of this equality, f_{-1} and f_n are the zero maps.)

Remark 0.0.1. For this exercise, you will need to brush up on notions such as:

1. Dimension of a vector space (e.g., number of elements in a basis) and
2. Quotient vector spaces,
3. Standard results in linear algebra. You may invoke them so long as you cite them.