3. Thought I'd end up with Sean

The word "next" is an example of a word that we have not rigorously defined, but we all probably have some intuition over.

3.1. **For positive integers.** If someone gives you some positive integer, is there a notion of a *next* largest positive integer?

(a) Write a concrete example of a positive integer a, and of the "next largest" positive integer to your example of a.
(b) Now, try to give a *definition* of what it means for a positive integer b to be the next largest positive integer to a. Concretely, this means: Provide a *test* that someone can perform on b to confirm that b is the "next" positive integer to a. Importantly, this test should be a test you can perform regardless of which positive integers a and b are presented to you.
(c) Tell me why your example fits your definition. (That is, tell me why your example passes your test.)

3.2. **For integers.** Does your definition make sense for integers that aren't necessarily positive?

(a) Write a concrete example of a non-positive integer a, and of the "next largest" integer to your example of a.
(b) Now, try to give a *definition* of what it means for an integer b to be the next largest integer to a.
(c) Tell me why your example fits your definition.

3.3. **For real numbers.** Now suppose that a is a real number.

(a) Do your definitions above make sense when a is a real number? In other words, given real numbers a and b, can you run your tests from above to see if b and a pass the test? (Caution: This is not asking whether the *word* "next" makes intuitive sense for your test, but it is asking whether the logical content of the definition makes sense—i.e., whether b and a can be tested as you demanded before.)
(b) Write a concrete example of a real number a that is not an integer. Using the definitions you came up with in the previous questions, can you write down an example of the "next largest" real number to your example of a?
(c) Look back at your past definitions of what it means for an integer b to be the next largest integer to a. Does your definition fit the "intuition" you'd have for the word *next*? Explain to me what you think, and why.
(d) Does a real number a have a "next largest" real number? Explain to me what you think, and why.

3.4. **For rational numbers.** Repeat the previous problem replacing every instance of the phrase "real number" with the phrase "rational number."

Remark 3.1. If you have time, try to convince yourself that every integer b is the "next" integer to some a.

Based on what you did in the previous problems, you may not be surprised if I tell you that a real number b is typically *not* the next real number to some real number a. However, every real number b *is* the "next" real number to some "increasing sequence" of real numbers $a_1, a_2, \ldots$. You'll learn more about this in analysis class; you may have even learned about it in Calculus II.

In fact, given a real number b, you can find some sequence $a_1, a_2, \ldots$ of *rational numbers* for which b is the "next" real number to this sequence! The fact that there is always a "next" real number to nice sequences of rational numbers, and that every real number is such a "next" real number, is actually the most important property of the set of real numbers. This is not how they teach it to you in previous math classes, but it's very true.

Indeed, the idea I'm getting at (if you want to Google it) is that the set $\mathbb{R}$ can be constructed from the set of rational numbers by adjoining all limits to increasing, bounded sequences.

Remark 3.2. The goal of this is to get you used to thinking about definitions as a *creative* process—not as something to be memorized. For example, I am willing to bet that our class can think of at least two different ways to define the notion of a "next" natural number.

Another goal is to set you up for induction proofs, which we will see in a few weeks.