4. Bigger and bigger sets

Definition 4.1. Let X and Y be sets, and let f be a function from X to Y. We will say that f is an *injection* if two distinct elements of X are never sent to the same element of Y.

There is a "mathy' way to write this. We say that f is an injection if the following holds: Whenver $x, x' \in X$ and $f(x) = f(x',)$, it must be that $x = x'$.

Here, the notation x' (read "x prime," just like in calculus) does not mean derivative, is just a lazy way of picking an element of X that may not necessarily be equal to x.

Notation 4.2. Let's write $X \leq Y$ if there exists an injection from X to Y. (This is a *new* meaning of the symbol $\leq$; note that X and Y are not numbers!)

(a) Using this meaning of $\leq$, explain that if $X \leq Y$ and $Y \leq Z$, then $X \leq Z$. (Here, Z is also a set.) (Hint: If f is a function from X to Y, and if g is a function from Y to Z, you can define a function from X to Z that sends $x \in X$ to $g(f(x))$.)
(b) Explain why any bijection from X to Y is also an injection from X to Y.
(c) Explain to me that if there exists a bijection from X to Y, then $X \leq Y$ and $Y \leq X$.
(d) Given a set X, let $\mathcal{P}(X)$ denote its power set. Show that $X \leq \mathcal{P}(X)$.
(e) Finally, give me an example of a set X and a set Y for which there is not a single injection from X to Y.