5. Having all the next ones

The goal of this homework assignment is to have you start thinking about some important properties of the set of natural numbers.

Recall that $\mathbb{N}$ is the set of all natural numbers. (In this class, this means $\mathbb{N}$ also includes 0.) As you know, there is a function from $\mathbb{N}$ to itself called "+1." Concretely, this function sends an element $n \in \mathbb{N}$ to $n + 1$.

We will call this function t.

Suppose there is some subset S of $\mathbb{N}$ which contains 0 and is closed under t. More precisely: Suppose there is some subset $S \subset \mathbb{N}$ such that (i) $0 \in S$ and (ii)

$$n \in S \qquad \text{implies} \qquad t(n) \in S.$$

(In other words, if we know that n is an element of S, then we may conclude that $n + 1$ is an element of S.)

Prompt One. Based on your knowledge of the natural numbers, do the above properties imply that S must equal $\mathbb{N}$?

Prompt Two. Now let $T \subset \mathbb{N}$ be an arbitrary, *non-empty* subset of $\mathbb{N}$. (This has nothing to do with Prompt One at face value.) Based on your knowledge of the natural numbers, must T have a smallest element?

Answer both prompts, one at a time.

As usual, for your homework submission, write a complete response, giving full reasoning. If you do not have an answer, you must explain what explorations you undertook—and you must write at least three appreciably different things that you tried to explore potential answers.