Lecture 1

Sets, and sets of numbers

Goals

1. (Terminology.) To understand what a set is, and what an element of a set is.

2. (Communicational.) To be able to see a drawing of a set and interpret the drawing. Conversely, to be able to draw certain kinds of sets.

3. (Notational.) To know what the symbols $\mathbb{R}$ and $\mathbb{N}$ mean.

4. (Mathematical.) To be able to identify whether a number is a real number, a natural number, or both, or neither.

5. (Awareness for future use.) To be aware that there is a set called the empty set.

1.1 What is a set?

Sets will form the foundation for the kind of mathematics we do in this course.

Here's a fact to put the importance of sets in context: It's probably safe to say that most of the mathematics you have done so far in your life came down to *numbers* in some sort of way. In this course, most of our mathematics will come down to *sets*.

Unfortunately, the notion of set will only be informally[1] defined:

Definition 1.1.1 (Informal)**.** A *set* is a collection of things.

Given a set, anything in that set is called an *element* of that set.

Remark 1.1.2. You can think of a set as a bag containing things; the elements of the set are the things in that bag.

Example 1.1.3. Here are some examples:

1. Choose three apples in your kitchen. We can speak of the set containing exactly those three apples; you might visualize this as a bag containing those three apples.

2. The things in a given set don't need to be the same "kind" of thing. For example, you can define a set to contain exactly one elephant, one shoestring, one pineapple, and one pebble. Such a set would contain four elements. If you want, you can visualize this set as a single bag containing an elephant, a shoestring, a pineapple, and a pebble. (It would be a very large bag.)

3. The things in a set don't even need to be physical objects. (In fact, for most of what we talk about, elements of a set will be quite abstract objects.) For example, think of a set consisting of three elements: "democracy," "friendship," and "Rick Astley." You can still imagine a bag, but perhaps it's harder to visualize how the bag would contain the very notion of democracy as an element. So we do not need to be able to visualize a set to understand that it can still be considered and talked about.

   This non-necessity of visualizing is a *feature* and an *advantage* of the abstract notion of set (really, the abstract notion of "thing" or "element"). If we could only speak of things we can draw or see, we would live in a very impoverished imagination.

[1] This is because it took mathematicians many modern decades of sophistication to try to rigorously define notions of set and of logic; it would take us a long time to explore this deep topic, and the exploration of this deep topic is (in my opinion) best appreciated only after gaining the other tools this course has to offer.

Also, it is often the case that for courses like ours, most students do not care too deeply about the intricacies of whether set theory is satisfactorily defined. So we will be content moving forward with an informal definition of sets. If you have no idea what I mean by whether something is satisfactorily defined, you can ignore this comment.

Example 1.1.4 (The empty set)**.** You can also consider the set with no elements. You might visualize this as a bag containing nothing. The set with no elements is called the *empty set*. This will come up over and over again.

Example 1.1.5. Here is another kind of set that we consider a lot in math: The set of all positive, even numbers. This set contains elements like 2, 4, 6, 8, 332, and so forth. This is our first example of a set with *infinitely* many elements in it. You probably have an intuitive or vague idea of what it means for a set to have infinitely elements; we will give this vague idea a definition later in this course.

Remark 1.1.6. The previous example illustrates the way we tend to define sets in mathematics. Let's break it down. We considered the set of "all positive, even numbers." Put another way, we considered the set of all numbers that satisfy the conditions of being even and being positive.

We will often begin with sets we know—like the set of all numbers—and then consider only certain things within that set—like only those numbers that are positive, and even.

Later, we'll see that we are considering *subsets* of a set we already know.

1.2 When are two sets the same?

Definition 1.2.1. We say that two sets are *equal*, or the *same* set, if they have the same elements.

Example 1.2.2. Consider a set containing the elements Bart, Lisa, and Homer.

If your friend considers a set containing the elements Bart, Lisa, and Homer, then your friend is considering the same set as we are.

Warning 1.2.3. Consider the set containing the elements Mom and Pop.

Consider a second set, containing the elements Door and Window.

Even though these two sets have the same number of elements (two), they are *not* the same set.

Figure 1.1: A set with three elements in it—Letter, Lancaster, and Lemonade—visualized as a bag containing these elements.

1.3 Drawing sets

There are many ways to draw sets. You could, if you wanted, draw a bag with things in it, as in Figure 1.1. Or, you could draw a set as a region—perhaps delineated by a circle or by a rectangle—with elements in it, as in Figures 1.2 and 1.3. Take a moment to look at all these figures. These figures are all drawings of the same set: the set containing three elements called Letter, Lancaster, and Lemonade. You may choose to draw the same set in many different ways. I won't care, nor will the set. Your drawing is your representation of the set.

As you look at these drawings, note that it doesn't matter how the elements are drawn inside the set—above, down, to the right, whatever—all that matters is that the elements are all contained in the set.

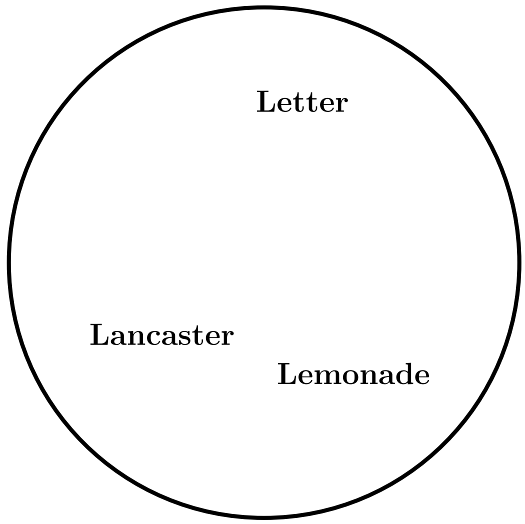


Figure 1.2: A set with three elements in it—Letter, Lancaster, and Lemonade—visualized as a round region containing these elements.

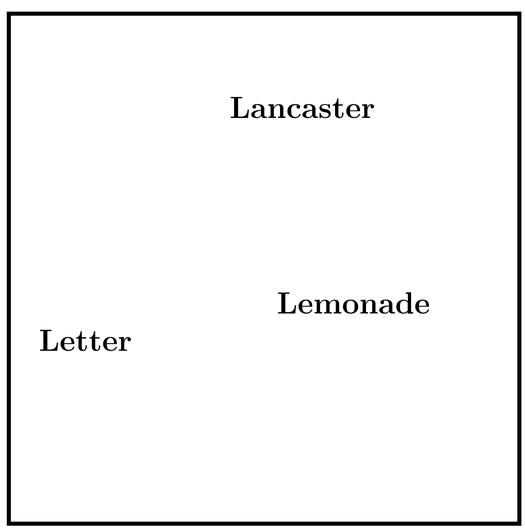


Figure 1.3: A set with three elements in it—Letter, Lancaster, and Lemonade—visualized as a rectangular region containing these elements.

Remark 1.3.1. Drawing sets isn't a skill that you'll be tested on; but it's a way of conceptualizing a set visually that may be helpful to you in the future.

If there is one conceptually important take-away you get from drawing sets, it's that things like the shape of the bag, the relative positioning of the elements, the shape of the region, do *not* matter to what the set actually is. All that matters about a set is its elements.

Warning 1.3.2. There are some sets that are impossible to draw in any meaningful way. So don't get too attached to the notion that you can draw sets; most sets in math, we have no way of drawing.

As an example, can you meaningfully draw the set of all hydrogen atoms in the universe? Whatever you attempt would be an abstract representation at best, and there's nothing wrong with that; so long as you remember that your drawing is a representation, and not a complete description/copy of the set.

With this in mind, it may be useful to still draw something which—while your drawing may not represent the set faithfully—helps you conceptualize what is happening in a visual way.

1.4 Sets of numbers

There are some sets that we use so often that we give them special symbols. As we will see, this use of these special symbols arose out of laziness, or out of a desire for efficiency. The symbols we'll learn are

$$\mathbb{R}, \qquad \mathbb{N}, \qquad \mathbb{Z}, \qquad \mathbb{Q}, \qquad \mathbb{C}$$

for the set of all real numbers, the set of all natural numbers, the set of all integers, the set of all rational numbers, and the set of all complex numbers, respectively.

Remark 1.4.1. $\mathbb{R}$ is the letter capital R, but drawn with extra lines. The font is sometimes called 'blackboard bold,' because on the blackboard, we tend to "bold" a letter just by adding some extra lines. Likewise, $\mathbb{Q}, \mathbb{Z}, \mathbb{N}, \mathbb{C}$ are the letters Q, Z, N, C but with blackboard bold font.

Everybody has a different way they like to write their blackboard bold fonts, but it is very important that you learn to write $\mathbb{R}$ in a way that distinguishes it from R. On a test, for example, it will be important that your handwriting distinguish between R and $\mathbb{R}$. Here are some ways to draw blackboard bold letters:

- $\mathbb{R}$ $\mathbb{R}$ $\mathbb{R}$
- $\mathbb{N}$ $\mathbb{N}$ $\mathbb{N}$
- $\mathbb{Z}$ $\mathbb{Z}$ $\mathbb{Z}$
- $\mathbb{Q}$ $\mathbb{Q}$ $\mathbb{Q}$
- $\mathbb{C}$ $\mathbb{C}$ $\mathbb{C}$

You should start practicing writing these letters in blackboard bold font—especially $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$—because they will come up over and over again.

Remark 1.4.2. It may seem very stupid that something as silly as a font is something you need to practice. But the blackboard bold convention for the above sets is deeply engrained into modern mathematical culture. (And, as I mentioned, part of the goals of this class is to help you enter the culture of modern research mathematics.)

Just as you probably learned to write x in a way that distinguishes it from the times symbol $\times$ (if you haven't, now is a good time!) it will aid you to learn to write $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}, \mathbb{C}$.

So let's dive into each of the above sets. For some of you, the facts below may be obvious or be tedious review; regardless, it is important for us as a class to begin on the same footing, so I want to spend some time making sure we are all on the same page.

1.4.1 $\mathbb{R}$

Notation 1.4.3. We let

$$\mathbb{R}$$

denote the set of all real numbers.

Example 1.4.4. Here are some examples of elements of $\mathbb{R}$:

$$0, \qquad 1, \qquad \pi, \qquad -e, \qquad \sqrt{2}, \qquad -105, \qquad \frac{2}{3}, \qquad \frac{\pi}{9}, \qquad \sqrt[9]{2}, \qquad 0.2,$$

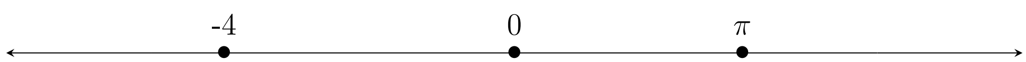
You can see it will be a fool's errand to try to write out every element of $\mathbb{R}$. (There are too many!)

Remark 1.4.5 (Ways to draw $\mathbb{R}$ and some of its elements)**.** You may have heard of "the number line." It is a common way people like to draw the set $\mathbb{R}$. Namely, some people draw $\mathbb{R}$ as a horizontal line:

The "dot dot dot" indicates that the line goes on forever. Sometimes people draw arrows to indicate this foreverness:

And either drawing is both common and acceptable.

Then, to draw an element of $\mathbb{R}$, people sometimes draw dots on the real line. Below, we have labeled three elements of $\mathbb{R}$:



Remark 1.4.6. One way that people like to think of $\mathbb{R}$ is as the set of "all possible decimal numbers." This depends on some prior knowledge of numbers, but typically, we are taught in elementary school to think of numbers using decimals. For example,

$$\pi = 3.1415926\ldots \qquad \frac{1}{3} = 0.333333\ldots, \qquad \text{two} = 2 = 2.000\ldots.$$

Then "any" number that we can write using decimals as above (while using possibly infinitely many places!) is a real number, and any real number has some decimal expansion.

Remark 1.4.7. Another way people like to think of $\mathbb{R}$ is as a "continuum." It is some collection of numbers with "no gaps." This is a very intuitive way of thinking about $\mathbb{R}$, but in a way that may not be obvious yet, it is quite hard to make this idea rigorous and useful for the purposes of *knowing* that certain facts about $\mathbb{R}$ are true.

We will most likely not have time to go into this subtlety. But a course that will develop a sophisticated way to think of $\mathbb{R}$ as a continuum is Real Analysis. And secretly, the deepest facts of calculus rely on such sophisticated thinking, but it is likely that your past calculus professors have kept these rich secrets from you.

1.4.2 $\mathbb{N}$

Definition 1.4.8. A *natural number* is a number that you can obtain from 0 by adding 1, perhaps many times.

Put another way, a natural number is a number that can be written as

$$0 + 1 + 1 + 1 + \ldots + 1.$$

Example 1.4.9. Here are some examples of natural numbers:

$$0, \qquad 1, \qquad 2, \qquad 3, \qquad 367, \qquad \ldots$$

You can see it will be a fool's errand to try to write out every natural number, because there are infinitely many.

Warning 1.4.10. Some textbooks do *not* think of 0 as a natural number. In our course, zero is a natural number.[2] This is simply a difference of convention. Don't worry, this isn't a mathematical issue. It is a cultural issue of the names we choose to give things, and it doesn't change the validity of mathematics.

Example 1.4.11. The number 0 can be obtained from 0 by adding 1 "no many" times. That is, by not adding 1 at all. This is why 0 is a natural number. Put another way, "some number of times" can include no number of times. This may sound wonky, but it's a kind of tricky reasoning that will show up over and over again.

Notation 1.4.12. We let

$$\mathbb{N}$$

denote the set of all natural numbers.

Remark 1.4.13. Here is another subtle thing about defining natural numbers: You have to know what 1 is, and what it means to add 1 to itself many times. Of course, everybody in this math class "knows" what these things are in their hearts; and you can be content with this. But imagine being asked the question: "Why is 1+1 equal to 2?" This is not a deep question in terms of understanding that it must be true; but it is a very deep question in understanding what "1+1" actually means, and what "2" actually means. More on this another day.

[2] I was raised to exclude 0, but I am going against my upbringing here.

1.4.3 $\mathbb{Z}$

Definition 1.4.14. An *integer* is a number you can obtain from 0 by either adding, or subtracting, 1 (perhaps many times).

Example 1.4.15. Every natural number is an integer, so numbers like 0, 1, 2, 13 are integers. Other examples of integers include $-3, -7, -1002$, et cetera. One might informally think of the difference between the set of natural numbers and the set of integers is that the latter allows for "negative versions" of natural numbers.

Remark 1.4.16. Why do we choose to use the letter Z? The German word for number is "zahl", or "zahlen" (for numbers).

1.4.4 $\mathbb{Q}$

You have probably heard of the term "rational number" before. There are two potentially conflicting definitions:

Definition 1.4.17 (Using division)**.** A number is called *rational* if it can be written as a quotient—i.e., a fraction—of two integers.

Example 1.4.18. So, by the above definition, we have the following examples of rational numbers:

$$\frac{1}{3}, \qquad \frac{5}{1}, \qquad \frac{-17}{23}, \qquad \frac{16}{4}, \qquad \frac{30}{35}.$$

Note that any integer is a rational number—for example, 5 can be written as $\frac{5}{1}$. Remember also the usual ways to equate fractions: $\frac{20}{30}$ is the *same* rational number as $\frac{2}{3}$, for example.

Here is another definition you may have seen:

Definition 1.4.19 (Using decimal expansions)**.** A decimal number is called a *rational number* if, after finitely many decimal places, the digits begin to repeat.

Example 1.4.20. For example, the number 5, which can be written as

$$5.000\ldots, \qquad \text{or} \qquad 5.\underline{0}$$

has a decimal expansion which clearly begins to repeat (by repeating the number 0). Another example is the number

$$0.142857142857142857142857142857\ldots, \qquad \text{aka} \qquad 0.\overline{142857}.$$

(This turns out to be equal to the number 1/7.) Note that the repeating string may be long—the example above has a string of 6 digits that repeats. Also, the decimal may not begin to repeat until many decimal places after the decimal point; that is okay. For example,

$$0.14567878787878\ldots = 0.145\overline{678}$$

is still a rational number by the above definition.

You will spend some time thinking about how these two definitions are related.

Notation 1.4.21. We let $\mathbb{Q}$ denote the set of all rational numbers.

1.4.5 $\mathbb{C}$

The following may not make much sense if you have not heard of $i = \sqrt{-1}$ before:

Definition 1.4.22. A *complex number* is any number of the form $a + bi$, where a and b are real numbers, and i is a square root of -1.

Notation 1.4.23. We let $\mathbb{C}$ denote the set of all complex numbers.

Remark 1.4.24. We can identify a real number as a special kind of complex number—a complex number for which $b = 0$.

1.5 Exercises

Exercise 1.5.1. Which of the following is an element of $\mathbb{R}$?

a. π

b. 0

c. 1

d. -3

e. e

f. $\frac{3}{4}$

g. $\sqrt{2}$

h. $5 + \sqrt{2}$

i. i, a square root of -1.

Exercise 1.5.2. Which of the following is an element of $\mathbb{N}$?

a. π	d. -3	g. $\sqrt{2}$
b. 0	e. e	h. $5+\sqrt{2}$
c. 1	f. $\frac{3}{4}$	i. i, a square root of -1.

Exercise 1.5.3. Which of the following is a true statement?

(a) If you take two natural numbers and subtract one from the other, the result is a natural number.

(b) If you take two real numbers and subtract one from the other, the result is a real number.

(c) If you take two natural numbers and multiply them, the result is a natural number.

(d) If you take two natural numbers and multiply them, the result is a real number.

(e) If you take two real numbers and multiply them, the result is a real number.

(f) If you take two non-zero natural numbers and divide one by the other, the result is a natural number.

(g) If you take two non-zero natural numbers and divide one by the other, the result is a real number.

(h) If you take two non-zero real numbers and divide one by the other, the result is a real number.

Exercise 1.5.4. Draw a set with five elements in it. Give your elements names so that it's easy to distinguish them. (As I've mentioned before, the way you draw a set is up to you; but make sure you draw it in a way that someone viewing it can identify the five elements and convince themselves that there are indeed five elements in your set.)