Lecture 2

Counting sizes of sets, and bijections

Goals

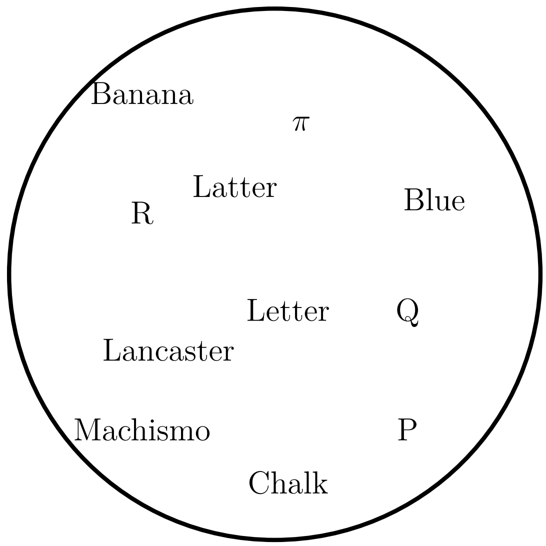
1. (Experiential.) To experience the exploratory, non-linear, messy process of trying to say "precisely" what a familiar concept is. The main example for us will be how to precisely describe counting.

2. (Intuitional.) To understand that we can determine the sizes of sets without needing an idea of "plus 1" or "next."

2.1 How do we know how many elements are in a set?

Recall that a set is a collection of things. And when somebody gives you a set, it's natural to ask: How many things are in that set?

Example 2.1.1. Below is a set:



How many elements are in this set?

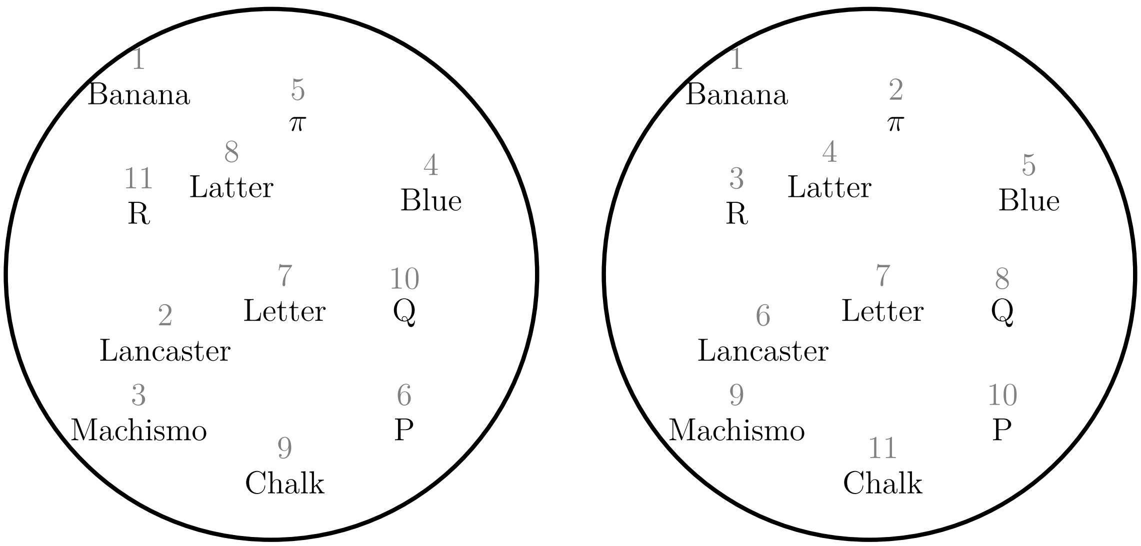
The answer to this question is probably uninteresting to you: There are eleven elements.

However, it is far more interesting to ask *how* we know there are eleven elements.

(Remember, one of the things we emphasize in this course is not just knowing that something is true, but knowing *why* something is true.)

So how did we know? We counted.

Yet, there are many ways in which we could count! Here are some examples of ways we might count the elements of this set:



(2.1.0.1)

And it turns out that, by examining how we know that the above set has 11 elements, we come upon a very important concept in mathematics: Bijections. Understanding what a bijection is is the main goal of today's class.

2.2 What is counting? When do we know a set has n elements?

When we count, it is very important that we do it carefully (so we get the correct answer!). So let's be specific about what counting is—what does it mean to count all the elements of a set?

Before we go on, whatever set we have in mind, let's give it a name: X. It's common in mathematics to give things names. That's okay. For now, let's say that we have a set, and to avoid writing "remember that set we have? The one with all the elemets?" all the time, we'll call the set X.

Attempt 2.2.1 (At understanding what counting is)**.** Suppose that X is a set with n elements.

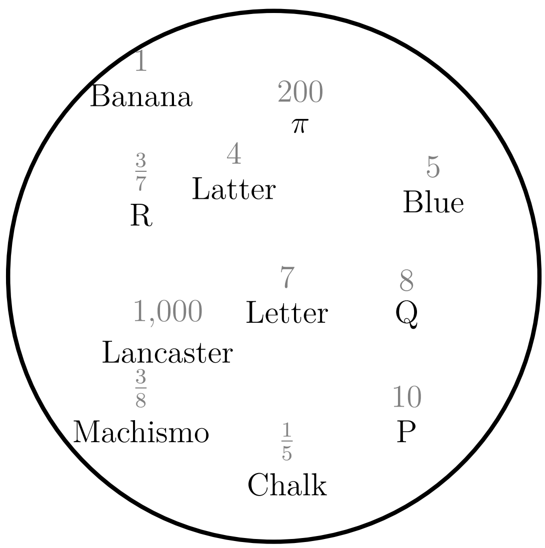
To "count" all the elements of a set X means to assign every element of X a number.

The above attempt at describing counting isn't quite watertight. For example, when we count, we do assign numbers to every element of a set, but we do so in a way that doesn't "reuse" numbers. We wouldn't count by saying "one, two, three, three, three, three, four, five, ..." So while the above attempt gets at describing counting, it doesn't quite describe it as accurately as we would like. Here is another attempt:

Attempt 2.2.2 (At understanding what counting is, Take Two)**.** Suppose that X is a set with n elements.

To "count" all the elements of a set X means to assign every element of X a number, and *without repeating numbers*[1].

Okay, we're getting there. But here's an assignment of numbers that fits the above description, but doesn't quite capture what we mean by counting:



(2.2.0.1)

Remark 2.2.3. What we're secretly doing is trying to tell an alien, or a robot, what exactly we mean by "counting all the elements" of a set X. You see, Attempt 2.2.4 didn't specify which numbers to use! So a silly alien could have come up with an assignment as in (2.2.0.1), which clearly is not the kind of assignment we have in mind when we count.

Attempt 2.2.4 (At understanding what counting is, Take Three)**.** Suppose that X is a set with n elements.

[1] In other words, two different elements of X are always assigned two different numbers.

To "count" all the elements of a set X means to assign every element of X a *natural number between 1 and* n, without repeating numbers[2], and *so that every number between 1 and* n *is used in the assignment.*

This is probably the most clunky way you would think of to describe counting, but I promise this does the job. In fact, I *so* promise that this is a good description of counting, let me call it a definition:

Definition 2.2.5 (Counting)**.** Let X be a set with n elements. A *counting* of all the elements of X is

- An assignment of a natural number between 1 and n to every element of X,

such that

(i) Two different elements of X are always assigned two different numbers, and

(ii) Every number between 1 and n is used in the assignment.

Here is a **very important** aspect of this discussion: You should now try to come up with examples of X, and of assignments that satisfy these conditions; then you should verify that each such assignment indeed seems to define a way to count the elements of X.

Explicit Advice 2.2.6. This is the part of learning math that is often left unsaid: When you see a definition of something, the best way to try to understand it is to produce examples and counter-examples to the definitions.

Example 2.2.7. Looking at Definition 2.2.5, you should not only produce examples of counting, but also examples of assignments that are *not* counting.

This then gives you a feel for what kinds of assignments really are counting, what kinds of assignments are not, and also, *why* I have chosen to call this idea a "counting."

[2] In other words, two different elements of X are always assigned two different numbers.

2.3 Counting defined differently

There might be a different way to think about counting. Instead of thinking of counting as "take an element of X and assign it a number," you may think of counting as "take a number and assign to it an element of X."

We'll explore *why* both ways of thinking work out later in this course. For now, let me present an alternative definition of counting:

Definition 2.3.1 (Counting, Take 2)**.** Let X be a set. Suppose we know it has n elements. Then a *counting* of X is:

- An assignment which, to every integer between 1 and n (inclusive), assigns an element of X

satisfying the following:

1. (You've counted everything.) The assignment hits every element of X. (That is, for any element of X, there is at least one number assigned to that element.)
2. (No double-counting.) Given an element of X, at most *one* number is assigned to that element.

2.4 Exercises

Exercise 2.4.1. Let X be the set containing the following elements: Rhinoceros, Batman, emptiness, hydrodynamics, and Shrek. (This set has five elements.)

(a) Exhibit three different countings of this set. (This means to produce three different examples that satisfy Definition 2.2.5.) Of course, it may be a bit awkward to say exactly what the assignment is. You might choose to make a drawing akin to (2.1.0.1).

(b) Give an example of an assignment—of elements of X to natural numbers between 1 and 5—that fail condition (i) of Definition 2.2.5.

(c) Give an example of an assignment—of elements of X to natural numbers between 1 and 5—that fail condition (ii) of Definition 2.2.5.

(d) Is it possible to give an example of an assignment from X to natural-numbers between 1 and 5 that fails (ii), but not (i) ?

(e) Is it possible to give an example of an assignment that fails (i), but not (ii)?

Exercise 2.4.2. Let X be the set containing the following elements: Fire, Light, and couch. (This set has three elements.)

How many different countings of X are there?

2.5 Some leftovers

2.5.1 Temptations and nextness

It is very natural to define counting in other ways. For example, you may very naturally try to define counting as: "Keep a running tally of numbers starting with 1. Each time you find a new element of X, add 1 to your running tally. When you have found all elements of X, your end tally tells you how many elements are in X."

This is a perfectly good definition. It does rely, however, on a notion of "plus one," or "next" number.

This does not mean that your thinking is incorrect. It does mean that we need the ingredient of "next" or "+1" to articulate your thinking. In class, I purposefully avoided making use of this notion. The notion of "next"ness turns out to be at the crux of how modern mathematicians even think about natural numbers.

2.5.2 How many ways to count are there?

Let's say that you know a set X has n elements. We saw in class that there are $n!$—n factorial—ways to count the elements of X. That is because there are n choices for assigning the number 1; $(n-1)$ choices for assigning the number 2, $(n-2)$ choices for assigning the number 3, and so forth. In all, we have

$$n \times (n-1) \times (n-2) \times \ldots 2 \times 1$$

choices for counting. This big number is called $n!$. We saw in class that the numbers of ways to count a set of $n = 14$ elements already exceeds the population of earth.

2.5.3 Function, domain, codomain

We also saw that, when we assigned numbers to elements of X, *every element of* X needed an assignment. Moreover, we also put the restriction that, the numbers we assign *must be a number from the target set* of numbers from 1 to n. These two italicized statements amount to saying that X is a *domain*, and the set of numbers from 1 to n is a *codomain*, of a function. This function is the assignment. You may be familiar with functions, domains, and codomains from calculus and precalculus. We will discuss this terminology further in future class sessions.

Remarks on your first homework

In your first homework, you were asked whether two definitions of the term "rational number" were equivalent: (a.) A number that can be expressed as a fraction of two integers, and (b.) A number whose decimal expansion eventually repeats some string of digits.

You will turn in a final draft of this assignment in a bit over a week. Here are some general comments on the assignments:

(1.) Dividing by zero. Almost everybody made a comment about "A fraction a/b is undefined if b equals zero." That is true. But this doesn't help with the problem, as it turns out.

And, I also believe there was some sort of group chat that influenced this thought process. Whoever proposed this idea, I want to give you encouragement. It was a natural idea to have, and that's important. In the future, keep proposing these ideas. The next step is for the whole group to think about whether a proposed idea is fruitful or not, rather than just people writing it out in their own assignments.

Let me explain why the idea of having zero in the denominator wasn't really relevant for this assignment.

For one thing, note that (a.) states "A number that can be expressed as..." In other words, first and foremost, a rational number is a number. So you already know a rational number is not some undefined thing. (It would be different if I said "A rational number is any expression a/b where a and be are integers." Then, you could complain that allowing b=0 is wonky.)

In other words, (a.) and (b.) are both specified to be descriptions of actual numbers, so stating that you could try to modify (a.) to include zero-denominator expressions would be addressing a different question altogether. You're modifying (a.) and using what might be called a "straw man" argument.

(2.) Some people tried to talk about 22/7 and pi. First, let's make clear that 22/7 and pi are two different numbers. So, the question is, how did people use these numbers to help them out? By and large, people did not.

Now, you could try computing 22/7 and see that it has a decimal expansion that repeats, so this might help you think through why fractions of integers might have repeating decimals. But doing this would be no more or less helpful than choosing a different fraction like 1/12 or 11/7.

Further, invoking pi, it turns out, doesn't help much. It is a very natural number to want to explore, but it turns out it is very very difficult to know whether pi satisfies either (a.) or (b.)—in fact, mathematicians didn't know for sure whether pi is irrational until around the founding of the United States. Consider that mathematicians knew about pi since at least the Babylonian and Egyptian civilizations (2000BC), so we couldn't get at the irrationality of pi for over 3000 years!

Along these lines: Many people claimed "pi is irrational." This is true. But there were two issues. (1) Nobody justified it (and I wouldn't expect us to be able to at this point—it is very difficult to prove that pi is not rational), and (2) Nobody used the irrationality of pi in a way that helped explore the assignment. For example, no matter how many digits of pi you write out, how do you know the digits won't eventually repeat?

(3.) Be warned: Examples typically are not enough. Many people (very correctly) wrote out some examples like $1/3 = 0.3333....$ But notice that examples don't establish that (a.) and (b.) are the same thing! Examples show that, in the examples you chose, (a.) and (b.) happen to be both satisfied.

If somebody told you, "You'll always win the lottery," and showed you three (and only three) examples of winning tickets, you wouldn't go buy a lottery ticket expecting to win. Those examples of lottery tickets that happened to have won don't prove anything!

However, examples are great places to begin *exploration*. If you believe you have found an example, you should explore *why* it is an example. For example, why really do you know that 1/12 has a repeating decimal expansion?

Finally, you might have tried to find a counter-example: that is, a number that satisfies (a) but not (b); or a number that satisfies (b) but not (a). Nobody was able to come up with a counter-example, by the way.

(4.) When you write, be honest, and don't make unjustified claims. You're allowed to say "I would like to claim that .... but I don't know how to see that this claim is true. Here's where I get stuck." Admitting that you don't know how to do something is totally okay, and in fact, being able to identify what you don't know is one of the most important parts of math and science.

A lot of people just stated facts without any justification. If you did so, you did not get any credit, and in fact, this typically hurt your overall score.

(5.) "Definition of rational." Many people seemed to use their own idea of what the word "rational" is supposed to mean. Some classmates used words like "correct definition." This isn't what the assignment asked for!

The assignment asked whether two proposed answers are the same. I could have said "Let's say that a number is super-duper if it satisfies (a.) Is satisfying (a.) the same thing as the number satisfying (b.)?" The particular words "super-duper" and "rational" are immaterial.

This isn't meant to tell you that you're doing it "wrong," but this is meant to tell you that you should make sure you know what the assignment is actually asking for.

(Now, as it turns out, and as you know, "rational" does have a definition. But knowing this definition won't help you complete the assignment! As I mentioned in class, the biggest resources you have are your ability to reason and explore; and your classmates.)

(6.) Finally, very few people tried to prove that (b.) implies (a.); in other words, that any decimal number with an eventually-repeating decimal expansion can be written as a fraction of two integers. I think only one person gave it a very nice try (and mostly got it, beyond some details).