Lecture 3

Bijections and sizes of sets

Goals

1. (Terminology.) To understand what a *bijection from* a set X *to* a set Y is.
2. (Conceptual.) To understand that a bijection is a way to tell that two sets have the same size, without using numbers.
3. (Conceptual.) To understand that counting is an example of a bijection.
4. (Communicational.) To be able to create drawings of simple assignments between sets.
5. (Awareness for future use.) To be aware that there are assignments that are not bijections.
6. (Mathematical.) To be able to verify whether certain assignments are bijections.

3.1 Knowing two sets have the same size without counting

We're going to keep talking about sizes of sets. Up to this point, we very tediously talked about what it means to know that X has n elements—that is, to count the elements of X.

Let's ask a different question. Suppose that we are given two sets, X and Y. How can we know they have the same size? That is, how do we know they have the same number of elements?

Of course, one possible answer is to count the elements of X, and count the elements of Y; if you end up with the same number, then we know that X and Y have the same number of elements.

But actually, there is a way to know whether X and Y have the same size *without* counting—and in fact, without using numbers at all!

Remark 3.1.1. I claimed that everything in this course will come down to sets (even though most of your previous math came down to numbers). The fact that we can tell that two sets have the same size *without* using numbers is one piece of evidence in support of my claim.

Example 3.1.2. Let's see examples of how to tell whether two sets have the same size, without counting.

1. Let X be the set of (cooked) hot dogs that Hiro has in his kitchen. Let Y be the set of hot dog buns in Hiro's kitchen.

   Then we can start assigning a bun to every hot dog. If every hot dog is assigned a bun, if each bun is paired up with no more than one hot dog, and no buns are left over, we know there are exactly as many hot dogs as there are buns.

2. Let X be the set of tutors at the Math Cats tutoring service at 3 PM, and let Y be the set of students there at 3 PM. Suppose that we can assign a tutor to every student so that (i) two different students are always assigned different tutors, and (ii) every tutor is assigned to some student.

   Then we know that there are exactly as many students as there are tutors!

3. Let X be the set of leading dancers in a ballroom dance class, and let Y be the set of following dancers. Suppose we can assign every leading dancer a following dancer so that (i) no two leading dancers are assigned the same following dancer, and (ii) every following dancer is assigned to some leading dancer. Then we know there are as many leads as there are follows.

As you can imagine, it is very important in math (and in life!) to know when two sets have the same size—i.e., the same number of elements. The power of the above three examples is that *assignments* allow us to avoid the counting, and hence the use of numbers. Isn't that amazing? *Assignments* let us avoid *numbers*.

Remark 3.1.3. In fact, the construction of numbers is not some "natural" inevitability of life, or of humanity. There are cultures in which there is no word for quantities like 3. For example the Hi'aiti'ihi people, otherwise known as the Pirahã, have words for one, for two, for few, and for many; but not for numbers like three, four, five, et cetera.

So being able to tell whether two sets have the same size—without counting—is an important task in cultures without the words to count!

(For an introduction to the connections between language and numerical cognition in the Pirahã, you can see "Numerical Cognition Without Words: Evidence from Amazonia" by Peter Gordon; this is an article published in 2004 in a very prestigious journal called *Science*. Make sure to read papers that reference this article to see how some people have come to analyze Gordon's findings in the years since.)

Another example of a language with "no big numbers" is the language of the Walpiri people. The Walpiri have words for one, two, and many (but not three, four, et cetera).

3.2 Bijections

The kind of assignments that help us avoid numbers is so important that we give it a name:

Definition 3.2.1 (Bijection)**.** Let X and Y be sets. A *bijection* from X to Y is:

- An assignment of an element of Y to every element of X,

such that

(i) Two different elements of X are always assigned two different elements of Y, and

(ii) Every element of Y is hit in the assignment.

Definition 3.2.2 (Bijection)**.** Let X and Y be sets. A *bijection* from X to Y is:

- A labeling, where we label every element of X with some element of Y,

satisfying the following conditions:

(i) (No double-labeling.) Two different elements of X are always assigned two different labels, and

(ii) (Every label is used.) Every element of Y is used as a label.

The main difference is that I used the idea of "labeling" instead of "assignment."

Remember, a "bijection" captures the following intuitive notion for now: A bijection is a way to see that two sets have the same size without counting their elements.

3.3 Bijections can be "inverted."

So assume you have a bijection from X to Y—that is, every element of X is labeled with an element of Y, in such a way that no label is used more than once, and every label is used.

Then, you can instead think of elements of X as labeling elements of Y! In other words, if you have a bijection from X to Y, then you automatically have a bijection from Y to X. This is called the *inverse* bijection of the original bijection.

Example 3.3.1. Suppose you can assign every hot dog in Hiro's kitchen a hot dog bun, in such a way that every hot dog bun is used up, and no two hot dogs are assigned the same bun. Well then, for every bun in the kitchen, you can ask: Which hot dog was assigned to this bun? The answer to this question defines the inverse bijection.

3.4 Counting is a bijection

I promised that we would discover the notion of a bijection when we thought deeply about counting. Let's dig in a bit more.

Notation 3.4.1. Let n be a natural number. If $n \geq 1$, we let

$$\underline{n}$$

denote the set containing all, and only, the natural numbers between 1 and n (including 1 and n).

If $n = 0$, we let $\underline{0}$ denote the empty set.

Remark 3.4.2. When you are reading the notation $\underline{n}$ out loud, I would suggest saying "underline n." So for example, $\underline{4}$ would be read "underline four."

Example 3.4.3. The set $\underline{3}$ has exactly three elements called 1, 2, and 3.

Then, in fact, to count the elements of a set X with n elements is the same thing as giving a bijection from X to $\underline{n}$. Think this over and make sure you understand why I say this. I leave it to you as Exercise 3.6.2.

So we have managed to discover an idea—a bijection—that not only allows us to describe the specific task of counting, but also allows us to relate sizes of sets without numbers!

3.5 Functions, domains, and codomains

We've used words like "assignment" and "labeling." The mathy term that most mathematicians use is "function." We previewed these terms last class, and now I am going to ask you to be able to use the following terms accurately:

Definition 3.5.1 (Function, domain, and codomain)**.** Let X and Y be sets. A *function* from X to Y is an assignment, assigning to every element of X some element of Y. Put another way, a function is a labeling of every element of X with some element of Y.

We say that X is the *domain* of the function. And we say that Y is the *codomain* of the function.

You may have also heard the term "target" instead of codomain. These are both commonly used terms for the function.

Remark 3.5.2. As we saw last class, there are many ways of making functions from X to Y. For example, when X was a set we wanted to count, and Y was the set of natural numbers from 1 to n, there were functions that "double-counted" (so for example, two different elements of X could be assigned two identical elements of Y) or "missed" elements of Y (so there was some element of Y that did not serve as any label).

The take-away is that *bijections are very special kinds of functions.*

For the next few classes, we won't talk about functions that aren't bijections; just be aware that there are such things. And don't worry: We will see plenty of functions that are not bijections as the semester progresses.

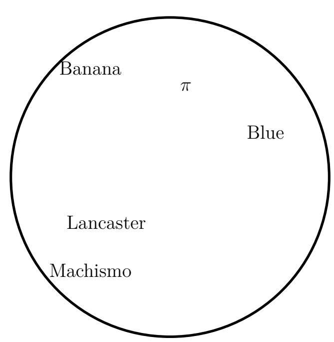
Remark 3.5.3. Using this terminology, suppose that you have a function from X to Y that is a bijection. Then you can define an *inverse* bijection, and this bijection is from Y to X; for the inverse bijection, the roles of domain and codomain are reversed. Y is now the domain, and X is the codomain.

3.6 Exercises

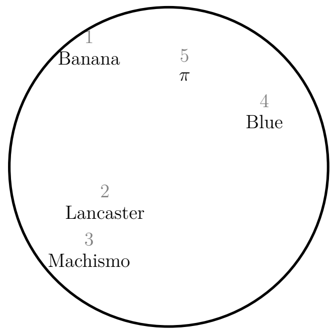
Exercise 3.6.1. Verify that the examples in Example 3.1.2 all describe bijections.

Exercise 3.6.2. Convince yourself that a count of the elements of a set with n elements (Definition 2.2.5) is the exact same thing as a bijection from X to $\underline{n}$.

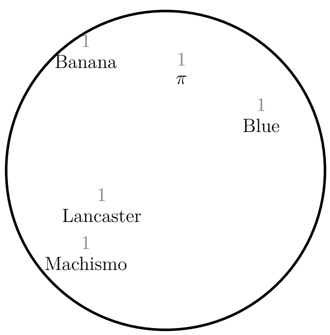
Exercise 3.6.3. Consider the set X consisting of the elements Banana, Lancaster, Machismo, Blue, and π. We draw X as follows:



Consider the assignment that assigns 1 to Banana, 2 to Lancaster, 3 to Machismo, 4 to Blue, and 5 to π. We draw this assignment as follows:



Which of the following assignments are bijections to $\underline{5}$?



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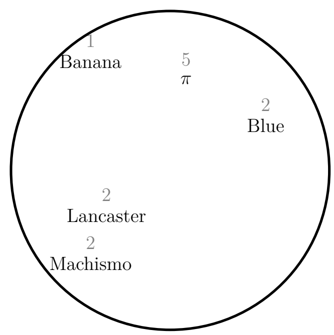

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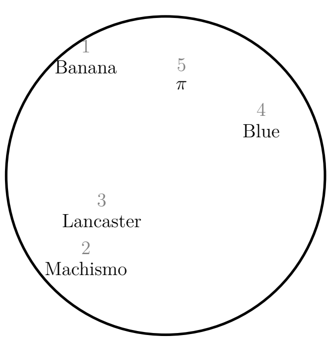

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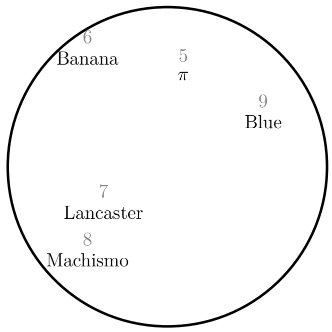

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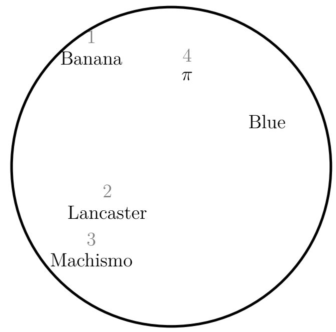

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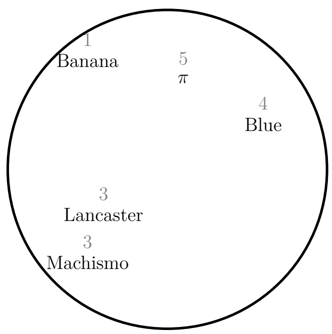

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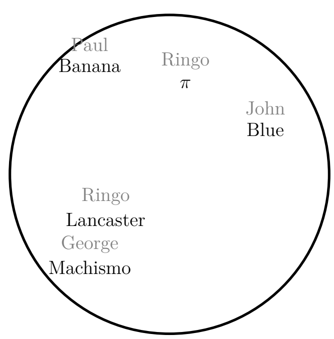
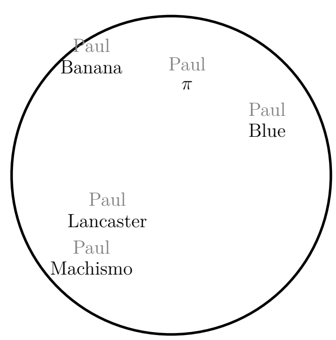
Exercise 3.6.4. Let X be the same set as the previous exercise. Let Y be the set containing the elements John, Paul, Ringo, and George.

Consider the assignment that assigns Paul to Banana, Ringo to Lancaster, George to Machismo, John to Blue, and Ringo to π. We draw this assignment as follows:



Which of the following assignments are bijections from X to Y?



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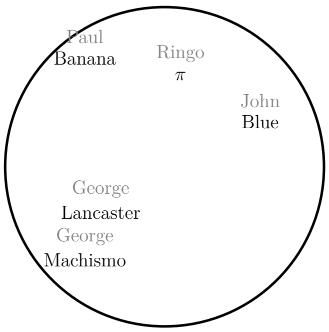

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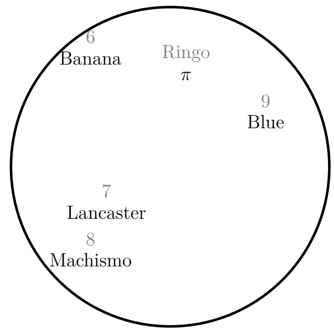

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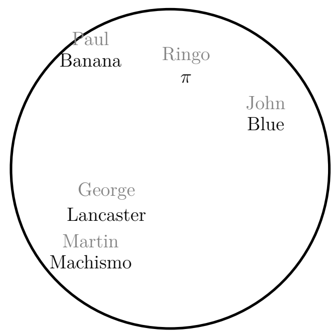

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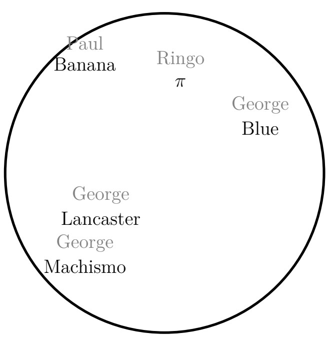

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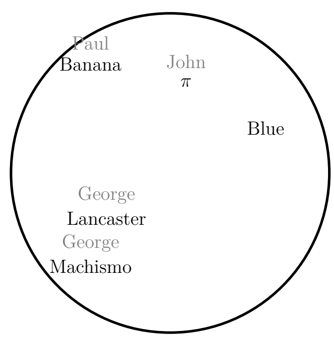

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Exercise 3.6.5. Let X and Y be two sets. Which of the following is true?

(a) The only way to tell if X and Y have the same size is to separately count the number of elements in each.

(b) You can tell whether X and Y have the same size without ever counting the number of elements in them.

(c) If you count the number of elements in X, and the number of elements in Y, and both counts result in the same number, then X and Y have the same size.

Exercise 3.6.6. Which of the following did Hiro say was a motivation for the topics from today's class?

(a) To kill time because Hiro couldn't wait for the weekend.

(b) To see that intuitive things can be hard to articulate precisely.

(c) To experience trying to give precise descriptions of ideas we're already familiar with.

(d) To build the ideas necessary to talk about sizes of *infinite* sets.

(e) To waste our time.

Exercise 3.6.7. Let X and Y be two sets. Suppose we know that X has exactly n elements in it, and Y has exactly m elements in it, where n and m are both natural numbers. Which of the following is true?

(a) If $n = m$, then there is a bijection between X and Y.

(b) If $n = m$, then there is exactly one bijection between X and Y.

(c) If $n \neq m$, then there is no bijection between X and Y.

(d) If there is a bijection between X and Y, then $n = m$.

(e) If there is a bijection between X and Y, it is still possible that $n \neq m$.

Exercise 3.6.8. Suppose that Hiro gives you a bijection from X to Y. Which of the following is true?

(a) If x and x' are two different elements of X, then Hiro's bijection assigns them two different elements of Y.

(b) To every element of X, Hiro's bijection assigns an element of Y.

(c) It is possible that some element of X is not assigned to any element of Y.

(d) It is possible that some element of Y does not receive an assignment from any element of X.

(e) It is possible that two different elements of X are assigned the same element of Y.

(f) It is possible that some element of X is assigned to two different elements of Y.

Exercise 3.6.9 (Exploring our intuitions.)**.** So, consider the following examples of X and of Y. Using your intuition, when do X and Y seem to have the same "size"?

(i) $X = \{\text{Alice, Bob, Jan}\}$ and $Y = \{7, \pi, q\}$.

(ii) $X = \mathbb{N}$ and $Y = \mathbb{Z}$.

(iii) $X = \mathbb{N}$ and $Y = \mathbb{Q}$.

(iv) $X = \mathbb{N}$ and $Y = \mathbb{R}$.

(v) $X = \mathbb{N}$ and Y is the set of all even, non-negative numbers. (0,2,4,6,8,10,12,14,...).

3.7 Our first flirtation with sizes of infinite sets

Recall that $\mathbb{N}$ is the set of all natural numbers. For us, this is the set of all numbers (including 0) that you can obtain by beginning with zero and adding 1. For example, $\mathbb{N}$ contains the number 0, 1, 2, 3, 4, 5, and so forth.

Let $X = \mathbb{N}$. Let Y be the set of all *positive* natural numbers, so the only difference from X is that Y does not contain 0.

Question. Is it possible that there is a bijection from X to Y?

Remember, this would intuitively mean that X and Y have the same "size."

Most sane humans believe that X is "bigger" than Y. After all, isn't it plain that X has one more element than Y, because X contains 0, but Y does not?

Here is a challenge, that I know we can solve in today's class: Find a bijection from X to Y.