Lecture 4

Subsets and power sets

Goals

1. (Terminology.) To understand what a subset of a set X is.

2. (Terminology.) To understand what the power set of a set X is.

3. (Conceptual.) To understand that one can always make a “bigger” set out of a set X.

4. (Conceptual.) To understand that if assuming a fact leads to a false result, the assumption must have been false. (Proof by contradiction.)

5. (Experiential.) To see that notation is meant to make communication more efficient.

6. (Experiential.) To see that reading and understanding math proofs can take a long time, and you are *supposed* to spend a long time on something to understand it.

In this course, we’ve talked about *bijections* as a way to tell when two sets have the same size.

One of our eventual goals is to understand that there are infinite sets of different sizes. This is a statement that most people have no intuition over, so we’ll be careful about developing it. Today’s lecture is one stepping stone in that direction; we’ll learn how to make new sets out of old.

4.1 The example from last time

So, let's let X be the set of all natural numbers. (Remember, for us, this means X also contains 0 as an element.) Let Y be the set of all positive real numbers (so Y does not contain 0).

Question: Could there be a bijection from X to Y (and vice versa)?

Last time, we saw that the "natural" or "obvious" way to think of the elements of X and Y was not a bijection. This "obvious" assignment would send the element 1 of Y to the element 1 of X, and likewise with the elements 2, 3, so forth.

We all saw that this would "leave out" the element 0 of X. So this assignment from Y to X would not be a bijection.

Remark 4.1.1. More intuitively, you might think of X as a set obtained from Y by "adding one more element" (namely, adjoining the element zero). And intuition might tell us that if we adjoin a new element to a set, of course we'd get a set with a different size. So of course there couldn't be a bijection between the two sets!

Your intuition would be correct in the world of *finite* sets. But, as we will see over time, one needs a different intuition in the world of the infinite.

For now, I won't say anything in the notes. Please keep thinking about whether X and Y might admit a bijection between them.

4.2 $\in$ Notation

We'll see a lot of new notions and notation today. Let me remind you (or tell you for the first time, if you haven't heard it before) that new notation is just a way for mathematicians to be lazy. For example, in calculus, the notation $f'(x)$ doesn't have any new mathematical ideas in it; it's just a stand-in for the idea of the derivative of f. And it is the *idea* that has mathematical content, not the notation.

Keeping that in mind, let's introduce:

Notation 4.2.1. Let A be a set. If a is an element of A, we write $a \in A$.

If a is not an element of A, we will write $a \notin A$.

Remark 4.2.2. When we see the symbols $x \in A$, we will often read it out loud as:

1. "x is an element of A" ,or

2. "x is in A", or–sometimes, depending on the surrounding sentence structure–

3. "x in A".

Example 4.2.3. Let X be a set with three elements: Tania, water, and circulation. Then Tania $\in X$, for example. Likewise, if we want to say "water is an element of X," it will be faster for us to write "water $\in X$" than to write out that whole phrase.

However, Roger $\notin X$.

Remark 4.2.4. Here's some history behind the symbol $\in$. It's actually derived from the greek letter ϵ—that is, lower-case "epsilon." You may have seen this Greek letter in calculus class when you learned about ϵ-δ proofs. This Greek letter is also written ε sometimes.

As far as I know, the first person to use $\in$ in this "is an element of" way was by the Italian mathematician Giuseppe Peano, in 1881. (His name will come up in the class again later—Peano was kind of a big deal.) At the time, ϵ stood for the Greek work $\epsilon\sigma\tau\iota$, "esti" (which means "is" in English). It's very likely that the way Peano thought of things, a set S wasn't just a collection of things, but typically a collection of things signifying some property. So for example, S might be the set of all *tall* humans. So to say $x \in S$ may have been conceived of as "x is tall," rather than as "x is an element of the set of tall humans." Hence ϵ may have literally meant "is" rather than "is an element of."

Another name to invoke is that of Richard Dedekind (his name often appears with Peano's name; they both came up with influential ideas); it may be that Dedekind used $\in$ first, and that Peano followed suit.

4.3 Subsets

We introduce the following term:

Definition 4.3.1 (Subset)**.** Let A and B be sets. We say that A is a *subset* of B if, whenever a is an element of A, a is also an element of B.

The term "subset" is hopefully intuitive, because the prefix "sub" invokes an intuition that you're probably used to. Regardless, let's see some examples:

Example 4.3.2. $\mathbb{N}$ is a subset of $\mathbb{R}$. This is because every natural number is a real number.

Example 4.3.3. Let X be the set containing a single element: John.

Let Y be the set containing five elements: John, Paul, Ringo, George, and Pete.

Finally, let Z be the set containing the elements John and Ralph.

Then X is a subset of Y, and X is also a subset of Z.

Note that Y is not a subset of X, nor of Z. Likewise, Z is not a subset of Y, nor a subset of X.

This is probably one of the most important examples:

Example 4.3.4. Let X be any set. Then the empty set is a subset of X. Put another way, the empty set is a subset of every set.

Remark 4.3.5. Why is this true? Well, recall that A is a subset of X if, for every element a of A, we can conclude that a is also an element of X. Put another way, A is a subset of X if every element of A passes the following test: Is the element in X as well?

Put another way, the only time A is *not* a subset of X is when A contains an element that is not an element of X.

And when A is the empty set, it contains no such element.

Remark 4.3.6. Here is another way to think about subsets of X. What sets can you make using only elements that are in X? Put another way, what sets are there that only contain elements that are in X? Such sets are the subsets of X.

Remark 4.3.7. If you like visualizations, here is another way to think about a subset. Think of X as a collection of things—perhaps you think of X as a giant goodie bag containing its elements. Then a subset of X is any goodie bag you can make by only choosing things from the original X.

Exercise 4.3.8. For each example of X below, write out every subset of X.

(a) X is the set containing exactly one element: Television.

(b) X is the set containing exactly two elements: Television and Radio.

(c) X is the empty set.

For each of the above, how many subsets does X have?

4.4 $\subset$ Notation

Here is the only other piece of notation I'd like you to learn today:

Notation 4.4.1 ($\subset$)**.** If A is a subset of B, we will write $A \subset B$. You can also write $B \supset A$.

Remark 4.4.2. When we see the symbols "$A \subset B$", we will often read it out loud as:

1. "A is a subset of B" or, depending on the surrounding sentence structure,

2. "A a subset of B."

Warning 4.4.3. Note that the notations $\in$ and $\subset$ mean *very, very* different things!

Example 4.4.4. Let A be the set containing just a single element called Feynman. Then the following statement is true:

$$\text{Feynman} \in A.$$

Likewise, the following statement is also true:

$$\text{The set consisting of one element called Feynman} \subset A.$$

However, the following statements are false:

$$\text{Feynman} \subset A,$$

$$\text{The set consisting of one element called Feynman} \in A.$$

4.5 The power set

In the previous exercise, you practiced the following process: Begin with a set X. Now write down all the subsets of X. In other words, you studied the *collection of all subsets of* X.

Remember that a collection of things is called a set. So in fact, the collection of all subsets of X is a new set. This set comes up so often we give it a name:

Definition 4.5.1. Let X be a set. Then the *power set of* X is the collection of all subsets of X.

Remark 4.5.2. You can think of the power set of X as a bag of bags. More precisely, the power set—being a set—can be thought of as a bag containing things (where each thing is an element of the set). So what things does this giant bag called the power set contain? It contains bags, which for sake of conversation one might think of as small bags. This is why I say that you can think of the power set as a bag of bags—i.e., as a giant bag containing smaller bags.

Our big goal will be to show that, regardless of X, the power set of X is never in bijection with X. In other words, the power set of X will never be the same size as X.

The reason this will be useful for us: We now have a way to "construct" sets of different sizes. Each time we have some set X, we can construct a new set of a different size: the power set of X.

I would like you to contemplate the following questions for next time:

1. If X is a set, is the power set of X a "bigger" set?
2. Take a look at the way you thought about the previous question. Does the way you thought about it make sense even if X is an infinite set?

4.6 Exercises

These exercises are for you to practice the notations of $\in$ and $\subset$, and the concepts of elements and subsets.

Exercise 4.6.1. Let A be the set consisting of two elements called Angelina and Andrea. Which of the following is true?

(a) Angelina $\in A$.

(b) Angelina $\subset A$.

(c) Andrea $\in A$.

(d) $\emptyset \subset A$

(e) Andrea $\subset A$.

Exercise 4.6.2. Let A be the set consisting of two elements called Angelina and Andrea. Let B be the set consisting of three elements called Angelina, Andrea, and Ana. Which of the following is true?

(a) $A \subset B$.

(b) $A \subset A$.

(c) A is a subset of B.

(d) $A \in B$.

(e) Angelina $\subset A$.

(f) Andrea $\in A$.

(g) Andrea $\in B$.

(h) $\emptyset \subset A$

(i) $\emptyset \subset B$

Exercise 4.6.3. Let A be the set consisting of two elements called Angelina and Andrea. Let B be the set consisting of three elements called Angelina, Andrea, and Ana. Which of the following is true?

(a) The power set of A contains four elements.

(b) The power set of B contains eight elements.

(c) The empty set is an element of the power set of A.

(d) B is an element of the power set of A.

(e) A is an element of the power set of A.

(f) A is an element of the power set of B.

(g) A is a subset of the power set of A.

Exercise 4.6.4. Let X be a set. Which of the following is a true statement?

(a) The power set of X is the set of all elements of X.

(b) X is the set of all elements of X.

(c) The power set of X is the set of all subsets of X.

(d) If there is a bijection from a set X to a set Y, then there is a bijection from the power set of X to the power set of Y.

Exercise 4.6.5. Let X be a finite set, and suppose X has n elements. (Here, n is a non-negative integer.) Based on the counting we've done so far, which of the following do we expect to be the size of the power set of X?

(a) n

(b) n^2

(c) 2^n

(d) $n! + 1$

(e) $n!$

(f) $\log(n)$.

Exercise 4.6.6. Let $X = Y = \mathbb{N}$. Define a function f from X to Y by declaring that, for every $x \in X$, $f(x) = x + 1$. Which of the following is true?

(a) If x is an element of X, then $f(x)$ is the element of Y to which f assigns x.

(b) If x is an element of X, then $x+1$ is the element of Y to which f assigns x.

(c) If x and x' are elements of X, and if $f(x) = f(x')$, then we know that $x = x'$.

(d) For every element y of Y, there is some $x \in X$ so that $f(x) = y$.

(e) f sends the element 3 of X to the element 9 of Y.

(f) f sends the element 3 of X to the element 3 of Y.

(g) f sends the element 3 of X to the element 4 of Y.

Exercise 4.6.7. Let f be a function from some set X to some set Y. Which of the following is true?

(a) If x is an element of X, then $f(x)$ is the element of Y to which f assigns x.

(b) If x and x' are elements of X, and if $f(x) = f(x')$, then we know that $x = x'$.

(c) For every element y of Y, there is some $x \in X$ so that $f(x) = y$.

Exercise 4.6.8. Let $X = Y = \mathbb{N}$. Define a function f from X to Y by declaring that, for every $x \in X$, $f(x) = x^2$. Which of the following is true?

(a) If x is an element of X, then $f(x)$ is the element of Y to which f assigns x.

(b) If x and x' are elements of X, and if $f(x) = f(x')$, then we know that $x = x'$.

(c) For every element y of Y, there is some $x \in X$ so that $f(x) = y$.

(d) f sends the element 3 of X to the element 9 of Y.

(e) f sends the element 3 of X to the element 3 of Y.

Exercise 4.6.9. Let $X = Y = \mathbb{R}$. Define a function f from X to Y by declaring that, for every $x \in X$, $f(x) = x^2$. Which of the following is true?

(a) If x is an element of X, then $f(x)$ is the element of Y to which f assigns x.

(b) If x and x' are elements of X, and if $f(x) = f(x')$, then we know that $x = x'$.

(c) For every element y of Y, there is some $x \in X$ so that $f(x) = y$.

(d) f sends the element 3 of X to the element 9 of Y.

(e) f sends the element 3 of X to the element 3 of Y.