Lecture 5

Are power sets bigger than the original sets?

You saw in Exercise 4.3.8 that, when X is a finite set, it seems like the power set of X is bigger than X.

So, do you think this is true when X is infinite?

We've been burned before. We've seen examples of two infinite sets where one seems bigger than another, yet there's a bijection between them! (In other words, when sets are infinite, sometimes our naive intuitions can be betrayed.)

5.1 Today's goal

Today, I would like you to understand that a set X can *never* have a bijection to its power set. Importantly, I would like your understanding to be robust enough to encapsulate examples of X when X is infinite.

As we've learned in this class, "counting" is not available to us when our sets are infinite. So we've had to take the "intuitive" idea of two sets having the same size and give it the following incarnation: Two sets have the same size if there is some *bijection* between them.

5.2 Preparation

At this point, you really should make sure you know what a bijection is. Your "intuitive" notion of "having the same size" will not help on its own. Go back to your notes—a bijection is a function satisfying two properties. You should know what those two properties are.

The reason you need to know this definition: If you don't know what it means for something to be BLAH, you can't prove any statement that has to do with something being BLAH.

Put another way, if you want to understand a statement that uses the notion of a bijection, you better know literally what a bijection is.

Likewise, you'll want to know what exactly the power set of X is.

5.3 Getting started

Suppose that there is a bijection from X to its power set. We'll call this bijection f. So, for every element $x \in X$, f assigns some subset $f(x) \subset X$.

Now, define $S \subset X$ to consist of those elements x for which

$$x \notin f(x).$$

What can you say?

5.4 Have fun

Okay, that's it. I promise that the most difficult ingredients of the understanding are on this page. In other words, if your task were to create a painting, all the paint has been given to you. Now I'm asking you to paint an understanding using these ingredients.

This is not easy. But I'll give you a hint: Now that we've defined S, and given that f is a bijection, what can you say?

At the end of the day, you will also have to use the following "obvious" fact:

Fact. Let W be a set, and a be anything. Then either $a \in W$, or $a \notin W$. Both cannot be true.

Good luck.