Lecture 6

More on functions

Goals

1. (Terminology.) To become more familiar with the terms element, set, function, domain, and codomian.
2. (Conceptual.) To understand that there is a wild zoo of functions out there.
3. (Experiential.) To take a definition, and begin creating examples of that definition.

So we've seen the idea of functions, mainly as a way to talk about bijections. (Remember, bijections are a way to "pair up" elements of a set X with elements of a set Y. "Bijection" is the only notion we have of detecting whether two sets have the same size *without* using numbers.)

But I think we need more practice, and more examples. So let's deepen our understanding of what functions are.

6.1 Functions

Fix two sets X and Y. What does it mean to give a function from X to Y?

The term function was defined in Definition 3.5.1.

Here are many equivalent ways to say what the word "function" means:

1. A function from X to Y is an assignment that assigns some element of Y to every element of X.

2. A function from X to Y is a labeling that labels every element of X is labeled by some element of X.

3. A function from X to Y is a process that takes any element of X as an input, and outputs an element of Y.

Notation 6.1.1. Just as we give sets names like X and Y, we will often give functions names. The most common kinds of function names are f and g.

Then, given an element called x of X, we let $f(x)$ denote the label that x is given by f. In other words, f is the assignment, and $f(x)$ is what x is assigned by the assignment. Keep in mind that $f(x)$ is an element of Y.

We call X the *domain* of the function. The important part here is not the letter X. What makes a set the domain is that the function labels the elements of the domain; that is, the inputs of the function are the elements of the domain.

We call Y the *codomain* of the function. This is the set of possible labels to use. It is the set of possible outputs.

6.2 Examples of functions

Example 6.2.1. Most examples familiar from calculus are functions where $X = Y = \mathbb{R}$. In other words, almost every function you've seen in calculus is a way to take an element of the set of real numbers, and output another real number. For example, here is a function, in words:

- If you give me a real number, I will output the real number obtained by squaring what you give me, and then adding 3.

This function, which can be described very accurately using words (we just did!) can also be written very succinctly if we give the function a name. Let's call the function h. Then we can write

$$h(x) = x^2 + 3.$$

The above equation says that h is a function which, whenever you input a real number called x, will output 3 more than the square of the input.

Remark 6.2.2 (Functions in calculus and algebra)**.** Algebra is *so* successful in its symbolism that—no matter how hard we try—professors find it difficult to actually tell students that the notation

$$f(x) = x^3 + 3x + \cos(x)$$

actually means.

Many students finish calculus courses with the feeling that functions as "just formulas."

This is not how we will view functions in this class.

However, every function you saw in calculus is an *example* of the idea of function we will see in class.

Example 6.2.3. So, let's see some non-calculus examples.

Let X be the set of students registered for this course. Let Y be the set of letters A, B, C, D, and F. (Yes... This is an ominous example.)

Then, at the end of this course, I will create a function from X to Y; this function might be called g for "grade." So, if there is a student called x, then $g(x)$ will be the grade they are assigned.

Remark 6.2.4. Note that multiple students could get the same grade.

Also, not every grade needs to be used—it's my hope that the element $F \in Y$ is not assigned by g!

So, depending on everybody's performance, it is possible that neither of the two properties of a bijection is satisfied. In particular, g may not be a bijection.

Regardless, every student must be assigned some grade. So g is a function from X to Y.

Example 6.2.5. Now let's instead let Q represent the set of real numbers between 0 and 100, inclusive. We'll let X and Y be as in the previous example.

Then a function from X to Q would be a way to assign, to every student in this course, a number between 0 and 100, inclusive. Let's call such a function s. Such a function s might show up at the end of this course when I have to assign every student a numerical score.

For example, if Joanna is a student in this course (so that Joanna $\in X$) then s(Joanna) is the score that Joanna receives at the end of the course. This score, because it must be an element of Q, will be some real number between 0 and 100.

On the other hand, let's define a function t from Q to Y as follows:

$$t(q) = \begin{cases} F & q \leq 59 \\ D & q \in (59, 69] \\ C & q \in (69, 79] \\ B & q \in (79, 89] \\ A & q \in (89, 100]. \end{cases}$$

Then, a reasonable way to think of the grade function g is as a composition of s and t. In other words, I might choose g to equal $t \circ s$.

Example 6.2.6. Let X be a set consisting of the elements 3 and π. Let Y be a set consisting of the elements 8, 9, and 10. Here are examples of functions from X to Y:

1. A function we'll call g, for which

$$g(3) = 8 \qquad \text{and} \qquad g(\pi) = 9$$

2. A function we'll call h, for which

$$h(3) = 9 \qquad \text{and} \qquad h(\pi) = 8$$

3. A function we'll call a, for which

$$a(3) = 10 \qquad \text{and} \qquad a(\pi) = 9$$

4. A function we'll call ϕ, for which

$$\phi(3) = 10 \qquad \text{and} \qquad \phi(\pi) = 10$$

and so forth.

6.3 Ways to visualize functions

There are at least two ways to make a visual representation of a function. One is to draw arrows from inputs to outputs; another is to make a table.

Both are very nice ways of conceptualizing functions; but they are typically only accurate for domains and codomains with finitely many elements.

We'll illustrate these ways via examples. All the following examples will be based on the possible "grade" function from Example 6.2.3.

We'll pretend that we only have seven students in the course: Diana, Dion, Ralph, Roma, Angelica, Buffy, and Carus. So X is a set with 7 elements.

A useful way to draw a function f is by drawing both the domain and the codomain as separate blobs.

Then, each time we have two elements $x \in X$ and $y \in Y$ for which $f(x) = y$, we draw an arrow from x to y.

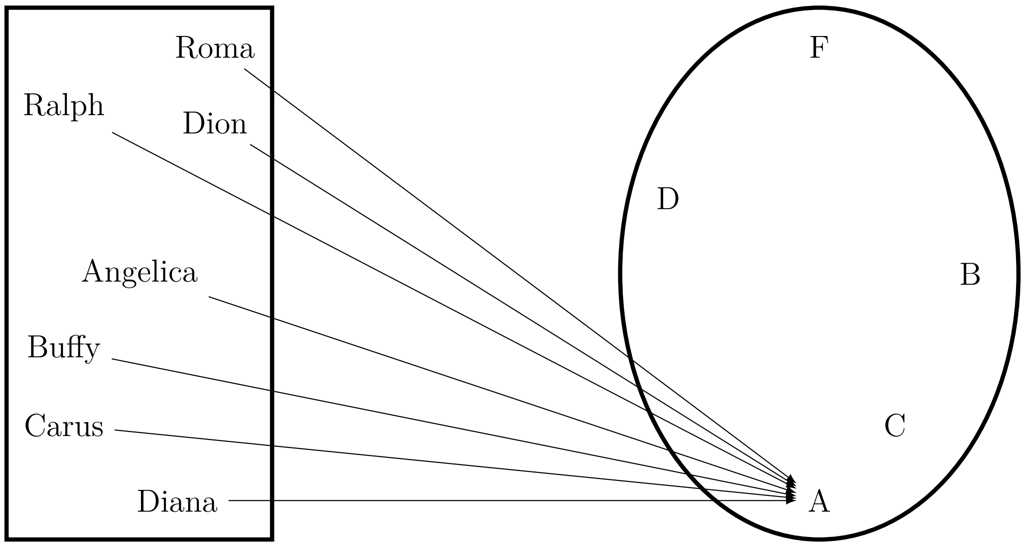

Figure 6.1: A function from X to Y with constant value A. This represents a grading function where "everybody gets an A." This depicts the same function as given in Table 6.1.

A	•	•	•	•	•	•	•
B							
C							
D							
F							
	Diana	Dion	Ralph	Roma	Angelica	Buffy	Carus

Table 6.1: The table representation of a function from X to Y which has constant value of $A \in Y$. This models a situation where everybody gets an A.

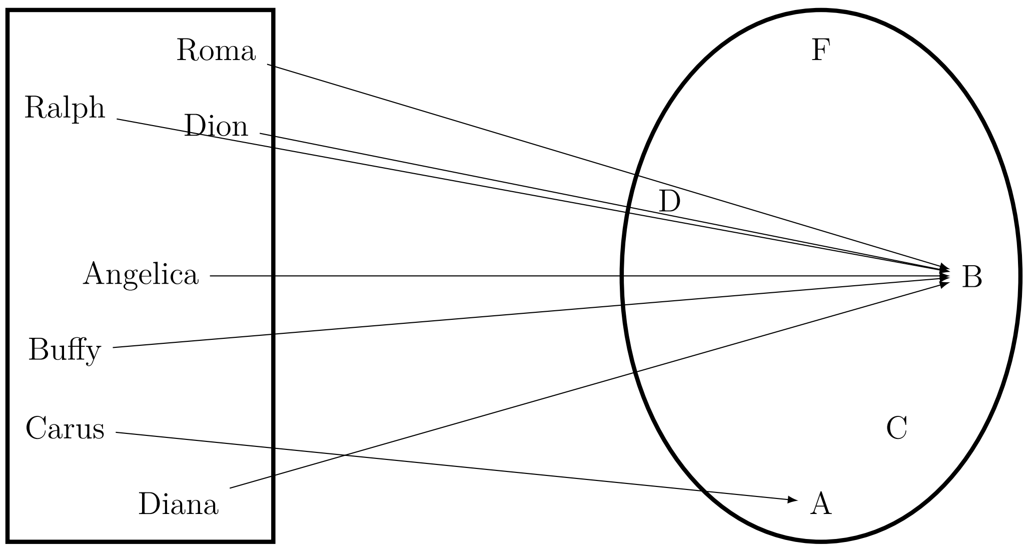

Figure 6.2: A function where everybody gets a B except Carus, who gets an A.

	Diana	Dion	Ralph	Roma	Angelica	Buffy	Carus
A							•
B	•	•	•	•	•	•	
C							
D							
F							

Table 6.2: A function from X to Y where everybody gets a B except for Carus, who gets an A.

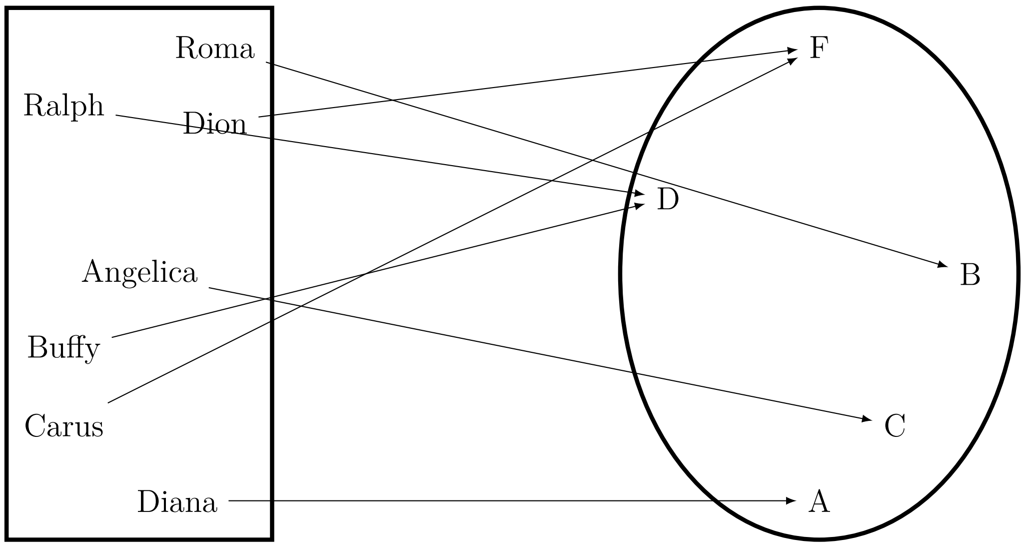

Figure 6.3: A function where every grade is used. See also Table 6.3.

	Diana	Dion	Ralph	Roma	Angelica	Buffy	Carus
A	•						
B				•			
C					•		
D			•			•	
F		•					•

Table 6.3: A function from X to Y where every element of Y is "used;" in other words, every available label is labeling something. This table represents a function which assigns A to Diana, F to Dion, D to Ralph, B to Roma, and so on. Note that the function represented here is not an injection—this is because two different elements of the domain (Carus and Dion) are sent to the same element F of the codomain. This represents the same function as in Figure 6.3.

6.4 Not every drawing is a function

Of course, we can make drawings out of functions; but not every drawing represents a function. In the following pages, we give two examples of drawings that are not functions. This may help illustrate what makes something into a function.

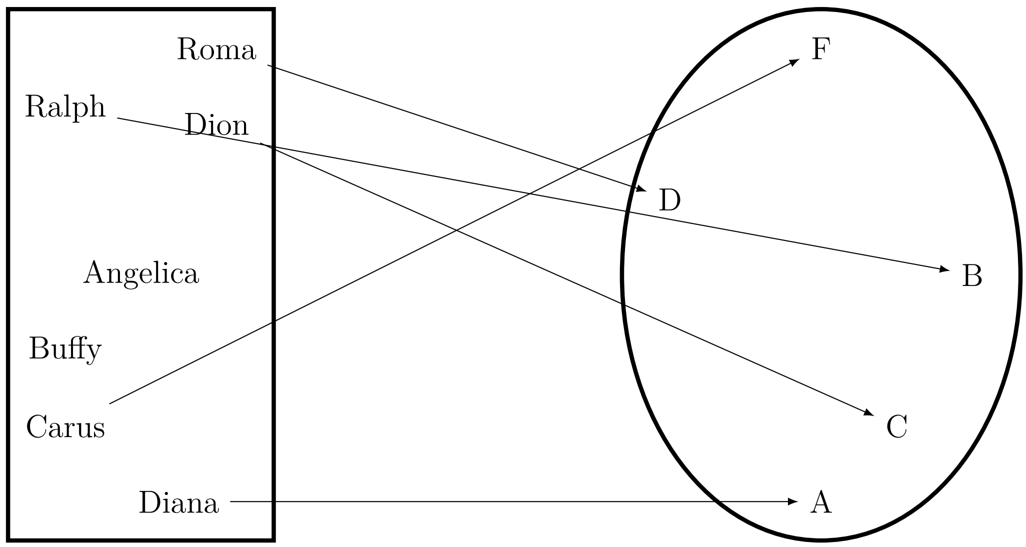

Figure 6.4: Not every way of drawing arrows from a domain to a codomain is a function. This picture does not represent a function from X to Y—the reason is that there are elements of X that are not assigned any value (but a function must assign a value to every element of the domain). Compare to Table 6.4.

	Diana	Dion	Ralph	Roma	Angelica	Buffy	Carus
A	•						
B			•				
C		•					
D				•			
F							•

Table 6.4: Not every way to fill in a table is a function. This table does not represent a function from X to Y, because a function from X to Y must tell you what to assign to every element of X. Here, we are not told what to assign to Angelica and Buffy. This table depicts the same data as depicted in Figure 6.4.

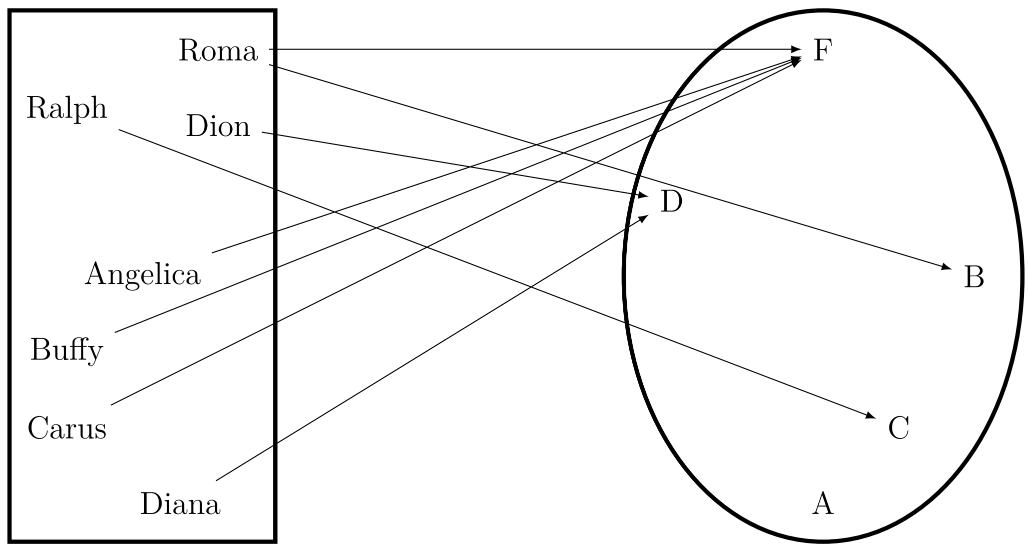

Figure 6.5: This diagram does not depict a function. The reason is that is an element (Roma) of the domain which is not assigned to a single element of the codomain, but which is assigned to two elements at once. Sometimes such an assignment is called a "multi-valued function;" for us, such an assignment is simply not a function. Compare to Table 6.5.

A							
B				•			
C			•				
D	•	•					
F				•	•	•	•
	Diana	Dion	Ralph	Roma	Angelica	Buffy	Carus

Table 6.5: Not every way to fill in a table is a function. This table does not represent a function from X to Y, because Roma is not assigned just on element of Y, but assigned both an A and an F. A function assigns to every element of X exactly one label (i.e., exactly one element of Y). This "only one label" property of functions is what gives meaning to the notation $f(\text{Roma})$, or more generally, $f(x)$ when $x \in X$—x receives exactly one label, and this label is $f(x)$. A visual way some people like to talk about this property is encoded in the "vertical line test." There is a vertical line through Roma's column that passes through two or more points, so this table cannot represent a function.

6.5 Special kinds of functions

So far, we have learned about two kinds of functions: bijections (in the first week of class) and injections (in homework).

The definitions below are equivalent to all the previous definitions of bijection and injection. You should make sure you understand why!

Definition 6.5.1 (Injection)**.** Let f be a function from X to Y. We say f is an *injection* if no two distinct elements of X are sent to the same element of Y.

Equivalently—and I encourage you to use the following formulation *only* if you fully understand the notation—we say that f is an injection if whenever $f(x) = f(x')$, then $x = x'$.

Definition 6.5.2 (Bijection)**.** Let f be a function from X to Y. We say f is a *bijection* if it is an injection and if every element of Y is assigned to some element of X.

Equivalently, f is a bijection if it is an injection and if, for every $y \in Y$, there exists some $x \in X$ such that $f(x) = y$.

Remark 6.5.3. The above are the cut and dry definitions of injections and bijections. We have talked about how a bijection, intuitively, is a pairing that matches up elements of X and elements of Y. But we must know what "pairing" means. The above definition gives the rigorous and precise meaning. Thus, these are the meanings you must use when discussing bijections in proof.