Solutions

7.1

Exercise 7.1.1. True

Exercise 7.1.2. True

Exercise 7.1.3. True

Exercise 7.1.4. False

Exercise 7.1.5. True

Exercise 7.1.6. True

Exercise 7.1.7. True

Exercise 7.1.8. True

Exercise 7.1.9. True

Exercise 7.1.10. False

Exercise 7.1.11. False

Exercise 7.1.12. True

Exercise 7.1.13. False

Exercise 7.1.14. True

Exercise 7.1.15. False

Exercise 7.1.16. True

Exercise 7.1.17. False

Exercise 7.1.18. True

Exercise 7.1.19. True

Exercise 7.1.20. False

Exercise 7.1.21. False

Exercise 7.1.22. False

Exercise 7.1.23. True

Exercise 7.1.24. True

Exercise 7.1.25. False

Exercise 7.1.26. True

Exercise 7.1.27. True

Exercise 7.1.28. True

Exercise 7.1.29. True

Exercise 7.1.30. False

Exercise 7.1.31. False

7.2

Exercise 7.2.1. False

Exercise 7.2.2. False

Exercise 7.2.3. True

Exercise 7.2.4. True

Exercise 7.2.5. False

Exercise 7.2.6. True

Exercise 7.2.7. True

Exercise 7.2.8. True

Exercise 7.2.9. True

Exercise 7.2.10. True

Exercise 7.2.11. True

Exercise 7.2.12. False

Exercise 7.2.13. True

Exercise 7.2.14. True

Exercise 7.2.15. False

Exercise 7.2.16. False

Exercise 7.2.17. False

Exercise 7.2.18. True

Exercise 7.2.19. True

Exercise 7.2.20. True

7.3

Exercise 7.3.1. False

Exercise 7.3.2. False

Exercise 7.3.3. True

Exercise 7.3.4. False

Exercise 7.3.5. True

Exercise 7.3.6. True

Exercise 7.3.7. True

7.4

Exercise 7.4.1. True

Exercise 7.4.2. True

Exercise 7.4.3. True

Exercise 7.4.4. True

Exercise 7.4.5. True

Exercise 7.4.6. True

Exercise 7.4.7. True

Exercise 7.4.8. True

Exercise 7.4.9. True

Exercise 7.4.10. False

Exercise 7.4.11. False

Exercise 7.4.12. False

Exercise 7.4.13. True

Exercise 7.4.14. True

7.5

Exercise 7.5.1. False

Exercise 7.5.2. False

Exercise 7.5.3. False

7.6

Exercise 7.6.1. True

Exercise 7.6.2. True

Exercise 7.6.3. True

Exercise 7.6.4. True

7.7

Exercise 7.7.1. True

Exercise 7.7.2. False

Exercise 7.7.3. False

Exercise 7.7.4. True

7.8

Exercise 7.8.1. False

Exercise 7.8.2. False

Exercise 7.8.3. False

7.9

Exercise 7.9.1. Let f be a bijection from X to Y. We define a function g from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ as follows: If A is a subset of X, declare $g(A)$ to be the subset of Y consisting exactly of the elements $f(x)$ with $x \in A$. In other words, an element y is in $g(A)$ exactly when there exists an element x in A such that $f(x) = y$.

I claim g is a bijection.

(1) First, we claim that g is an injection. So suppose $g(A) = g(B)$. This implies that $g(A) \subset g(B)$ and $g(B) \subset g(A)$.

(i) If $y \in g(A)$, this means there exists $x \in A$ for which $y = f(x)$ (this is by definition of $g(A)$). Because f is an injection, there is only one such x; so if y is also in $g(B)$, we see that x is also in B. We have shown that if $g(A) \subset g(B)$, then $A \subset B$.

(ii) By a symmetric argument swapping the roles of A and B, we see that $g(B) \subset g(A)$ implies $B \subset A$.

Combining (i) and (ii), we see that $g(A) = g(B)$ implies $A = B$. In other words, g is an injection. This finishes (1).

To complete our proof that g is a bijection, we must now prove that (2) for every element Z in $\mathcal{P}(Y)$, there exists an element $W \in \mathcal{P}(X)$ for which

$g(W) = Z$. To see this, let W be the set of all $x \in X$ for which $f(x) \in Z$. I claim that $g(W) = Z$.

(iii) First, let us show $g(W) \subset Z$. Well, by definition of W, we know that if $x \in W$ then $f(x) \in Z$. In particular, by definition of g, any element in $g(W)$ is also an element of Z. This shows $g(W) \subset Z$.

(iv) Now let us show $Z \subset g(W)$. This means we must show that if $y \in Z$, then $y \in g(W)$ as well. Well, given y, we may use the fact that f is bijection to conclude there exists $x \in X$ for which $f(x) = y$. By definition of W, this means $Z \subset g(W)$.

Combining (iii) and (iv), we see that $g(W) = Z$. In other words, for every $Z \in \mathcal{P}(Y)$, we have produced $W \in \mathcal{P}(X)$ for which $g(W) = Z$. This finishes (2).

By combining (1) and (2), we see that g is a bijection.

Exercise 7.9.2. Let us define a function f from X to $\mathcal{P}(X)$ as follows: Given $x \in X$, we declare $f(x)$ to be the set consisting of exactly one element, x itself.

If you suppose $f(x) = f(x')$, then we know that $f(x)$ and $f(x')$ must be the same set. By definition of f, $f(x)$ set must be the set containing only one element—x—and $f(x')$ is the set containing only one element—x'. So for these two sets to be equal, it must be that $x = x'$. To summarize, we've shown that if $f(x) = f(x')$, then $x = x'$.

So this proves f is an injection from X to $\mathcal{P}(X)$.

Exercise 7.9.3. Let $W \in \mathcal{P}(A)$. Then W is a subset of A—this means every element of W is an element of A. Because $A \subset B$, we know that every element of A is an element of B. Combining the previous two sentences, we see that every element of W is an element of B. In other words, $W \subset B$; that is, $W \in \mathcal{P}(B)$.

We have shown that $W \in \mathcal{P}(A)$ implies $W \in \mathcal{P}(B)$. This shows $\mathcal{P}(A) \subset \mathcal{P}(B)$.

Exercise 7.9.4. There are many ways to do this, but none of them are easy.

Here is one thing you can do. An extra credit assignment claims that there is a bijection from $\mathcal{P}(\mathbb{N})$ to the set of all functions from $\mathbb{N}$ to the set consisting of two elements—0 and 1. Let's take this for granted.

Now, given a function f from $\mathbb{N}$ to the set consisting of only 0 and 1, we can define a decimal number as follows: Declare its $2n$th decimal place to be $f(n)$, and all other decimal places to be 0. So for example, if f were a function taking 0 to 1, 1 to 1, 2 to 1, 3 to 1, 4 to 0, and 5 to 1 (I haven't specified what f does to other elements of $\mathbb{N}$) then the decimal number associated to f would be written as:

$$1.0101010001.....$$

We will call the number associated to f $\alpha(f)$. So α defines a function from "the set of all functions from $\mathbb{N}$ to the set consisting only of 0 and 1" to $\mathbb{R}$.

I claim α is an injection.

To prove α is an injection, we must show that if $\alpha(f) = \alpha(g)$, then $f = g$.

If $\alpha(f) = \alpha(g)$, then we know that $\alpha(f) - \alpha(g) = 0$. (Note subtraction makes sense because $\alpha(f)$ and $\alpha(g)$ are real numbers!) Now, by the ways that subtraction of real numbers works, and because $\alpha(f)$ and $\alpha(g)$ only consists of 0s and 1s (and no 9s) this turns out to imply that $\alpha(f)$ and $\alpha(g)$ must have exactly the same digits—that is, the 0th place must agree, the 1st decimal place must agree, the 2nd decimal place agree, and so forth. In other words, f and g assigns, to every element of $\mathbb{N}$, agreeing values. (Put another way: For every $n \in \mathbb{N}$, we now know $f(n) = g(n)$.) This shows $f = g$.

In summary, we have shown that if $\alpha(f) = \alpha(g)$, then $f = g$. In other words, α is an injection.

Now, let's recall our homework claim from earlier. We have a bijection (in particular, an injectino) from $\mathcal{P}(\mathbb{N})$ to the set of functions from $\mathbb{N}$ to the two-element set consisting of 0 and 1. We have an injection α from this set of functions to $\mathbb{R}$. Another part of previous homework is that the composition of two injections is an injection—thus, we have exhibited an injection from $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$.