Lecture 7

Review questions

For each of the following statements, indicate whether the statement is true or false.

7.1 Elements, subsets

Exercise 7.1.1. For any set X, the empty set is a subset of X.

Exercise 7.1.2. $\emptyset$ is a symbol for the empty set.

Exercise 7.1.3. If X is a set, then $\emptyset \subset X$.

Exercise 7.1.4. If X is a set, then $\emptyset \in X$.

Exercise 7.1.5. If X is a set and $\mathcal{P}(X)$ is its power set, then $\emptyset \in \mathcal{P}(X)$.

Exercise 7.1.6. If X is a set and $\mathcal{P}(X)$ is its power set, then $\emptyset \subset \mathcal{P}(X)$.

Exercise 7.1.7. The empty set contains no elements.

Exercise 7.1.8. If a set X contains at least one element, then X cannot be the empty set.

Exercise 7.1.9. Let X be a set. Then the power set of X is the set of all subsets of X.

Exercise 7.1.10. Let X be a set. If $\mathcal{P}(X)$ is the power set of X, and if $x \in X$, then $x \in \mathcal{P}(X)$.

Exercise 7.1.11. Let X be a set. If $\mathcal{P}(X)$ is the power set of X, and if $x \in X$, then $x \subset \mathcal{P}(X)$.

Exercise 7.1.12. Let X be a set. If $\mathcal{P}(X)$ is the power set of X, and if $A \subset X$, then $A \in \mathcal{P}(X)$.

Exercise 7.1.13. Let X be a set. If $\mathcal{P}(X)$ is the power set of X, and if $A \subset X$, then $A \subset \mathcal{P}(X)$.

Exercise 7.1.14. Let X be a set. If x is an element of X, then $x \in X$.

Exercise 7.1.15. Let X be a set. If x is an element of X, then $x \subset X$.

Exercise 7.1.16. Let X be a set. If $\mathcal{P}(X)$ is the power set of X, and if B is a subset of X, then B is an element of $\mathcal{P}(X)$.

Exercise 7.1.17. Let X be a set. If $\mathcal{P}(X)$ is the power set of X, and if B is a subset of X, then B is a subset of $\mathcal{P}(X)$.

Exercise 7.1.18. Let W and Z be sets. If $W = Z$, then $W \subset Z$.

Exercise 7.1.19. Let W and Z be sets. If $W = Z$, then $Z \subset W$.

Exercise 7.1.20. Let W and Z be sets. If W does not equal Z, then we know that $W \not\subset Z$.

Exercise 7.1.21. Let W and Z be sets. If W does not equal Z, then we know that $W \subset Z$.

Exercise 7.1.22. Let W and Z be sets. If $W \subset Z$, then $W = Z$.

Exercise 7.1.23. Let W and Z be sets. If $W \subset Z$ and $Z \subset W$, then $W = Z$.

Exercise 7.1.24. Let W and Z be sets. If $W \subset Z$ and $Z \subset W$, then W and Z are the same set.

Exercise 7.1.25. $\mathcal{P}(\emptyset) = \emptyset$.

Exercise 7.1.26. $\mathcal{P}(\emptyset) \supset \emptyset$.

Exercise 7.1.27. $\emptyset \subset \mathcal{P}(\emptyset)$.

Exercise 7.1.28. $\emptyset \in \mathcal{P}(\emptyset)$.

Exercise 7.1.29. $\mathcal{P}(\emptyset)$ has exactly one element.

Exercise 7.1.30. $\mathcal{P}(\emptyset)$ contains no elements.

Exercise 7.1.31. $\mathcal{P}(\emptyset)$ contains exactly two elements.

7.2 Functions

Exercise 7.2.1. Let X and Y be sets. If f is a function from X to Y, then f is a bijection.

Exercise 7.2.2. Let X and Y be sets. If f is a function from X to Y, then f is an injection.

Exercise 7.2.3. Let X and Y be sets. If f is a bijection from X to Y, then f is a function from X to Y.

Exercise 7.2.4. Let X and Y be sets. If f is an injection from X to Y, then f is a function from X to Y.

Exercise 7.2.5. Let X and Y be sets. If f is an injection from X to Y, then f is an bijection.

Exercise 7.2.6. Let X and Y be sets. If f is a bijection from X to Y, then f is an injection.

Exercise 7.2.7. Let X and Y be sets. If there exists a bijection from Y to X, then there exists a bijection from X to Y.

Exercise 7.2.8. Let X and Y be sets. If there exists an injection from Y to X, then there exists an injection from X to Y.

Exercise 7.2.9. Let X, Y, and Z be sets. If there exists an injection from X to Y, and if there exists an injection from Y to Z, then there exists an injection from X to Z.

Exercise 7.2.10. Let X, Y, and Z be sets. If there exists a bijection from X to Y, and if there exists a bijection from Y to Z, then there exists a bijection from X to Z.

Exercise 7.2.11. Let f be a function from X to Y. If there are two elements x, x' in X with $x \neq x'$ for which $f(x) = f(x')$, then f is not an injection.

Exercise 7.2.12. Let f be a function from X to Y. If there are two elements x, x' in X with $x \neq x'$ for which $f(x) \neq f(x')$, then f is not an injection.

Exercise 7.2.13. Let f be a function from X to Y. If for every pair of elements x, x' in X with $x \neq x'$, we know that $f(x) \neq f(x')$, then f is an injection.

Exercise 7.2.14. Let f be a function from X to Y. If there exist two distinct elements of X that are sent to the same element under f, then f is not an injection.

Exercise 7.2.15. Let f be a function from X to Y. If there exist two distinct elements of X that are not sent to the same element under f, then f is an injection.

Exercise 7.2.16. Let f be a function from X to Y. If for every $y \in Y$, there exists $x \in X$ for which $f(x) = y$, then f is a bijection.

Exercise 7.2.17. Let f be a function from X to Y. If for every element of Y, there is some element of X sent to that element by f, then f is a bijection.

Exercise 7.2.18. Let f be an injection from X to Y. If for every $y \in Y$, there exists $x \in X$ for which $f(x) = y$, then f is a bijection.

Exercise 7.2.19. Let f be an injection from X to Y. If for every element of Y, there is some element of X sent to that element by f, then f is a bijection.

Exercise 7.2.20. Let f be a function from X to Y and let $x \in X$. Then $f(x)$ is the element of Y that f assigns to x.

7.3 Combining notation

Exercise 7.3.1. Let X and Y be sets. If there exists an injection from X to Y, then $X \subset Y$.

Exercise 7.3.2. Let X and Y be sets. If there exists a bijection from X to Y, then $X = Y$.

Exercise 7.3.3. Let f be a function from X to Y. Then for any $x \in X$, we know that $f(x) \in Y$.

Exercise 7.3.4. Let f be a function from X to Y. Then for any $y \in Y$, we know there exists $x \in X$ for which $f(x) = y$.

Exercise 7.3.5. Let f be an injection from X to Y. Then whenever $f(x) = f(x')$, we know that $x = x'$.

Exercise 7.3.6. Let f be a function from X to Y. Suppose we know that whenever $f(x) = f(x')$, we must have $x = x'$. Then f is an injection.

Exercise 7.3.7. Let f be a bijection from X to Y. Then for every $y \in Y$, define $g(y)$ to be the element $x \in X$ for which $f(x) = y$. Then g is a bijection from Y to X.

7.4 Some examples

Exercise 7.4.1. Let X and Y be finite sets. If both X and Y have n elements, then there exists a bijection from X to Y.

Exercise 7.4.2. Let X be the set of all natural numbers, and let Y be the set of all positive natural numbers. Then there exists a bijection from X to Y.

Exercise 7.4.3. Let X be the set of all natural numbers, and let Y be the set of all positive natural numbers. Then there exists an injection from X to Y.

Exercise 7.4.4. Let X be the set of all natural numbers, and let Y be the set of all positive natural numbers. Then there exists a bijection from Y to X.

Exercise 7.4.5. Let X be the set of all natural numbers, and let Y be the set of all positive natural numbers. Then there exists an injection from Y to X.

Exercise 7.4.6. For any set X, there exists an injection from $\emptyset$ to X.

Exercise 7.4.7. If $A \subset B$, then there exists an injection from A to B.

Exercise 7.4.8. If X is a set, then there exists an injection from X to $\mathcal{P}(X)$.

Exercise 7.4.9. Let n be a natural number. If X has n elements, and if Y has $n+1$ elements, then there exists an injection from X to Y.

Exercise 7.4.10. Let n be a natural number. If X has n elements, and if Y has $n+1$ elements, then there exists a bijection from X to Y.

Exercise 7.4.11. Let n be a natural number. If X has n elements, and if Y has $n+1$ elements, then there exists an injection from Y to X.

Exercise 7.4.12. Let X be a set with exactly two elements called Ana and Paper. Then every subset of X contains the element Ana.

Exercise 7.4.13. Let X be a set with exactly two elements called Ana and Paper. Then there exists a subset of X that contains the element Ana.

Exercise 7.4.14. Let X be a set with exactly two elements called Ana and Paper. Then there exists a subset of X that does not contain the element Ana.

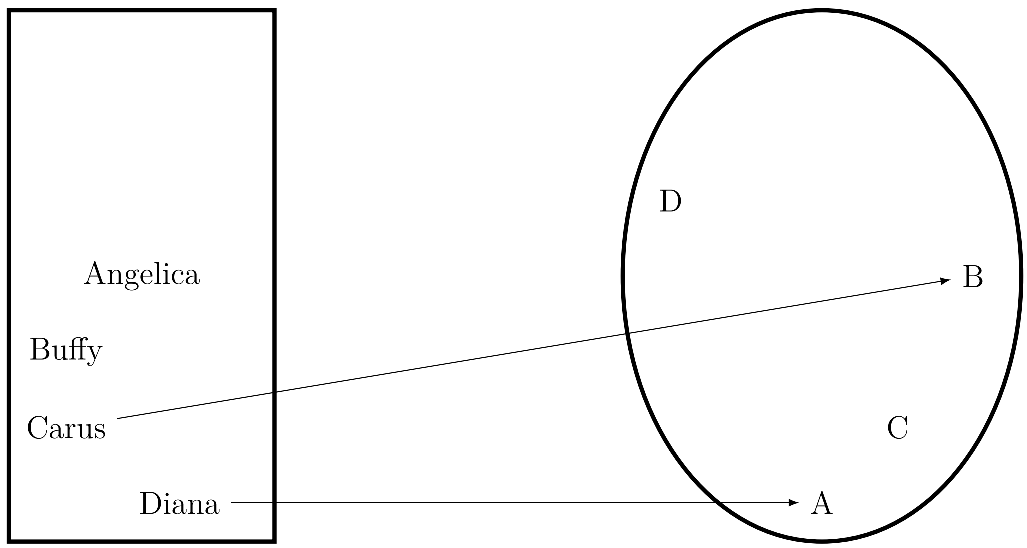

Figure 7.1:

7.5 Picture One

Exercise 7.5.1. Figure 7.1 depicts a function.

Exercise 7.5.2. Figure 7.1 depicts an injection.

Exercise 7.5.3. Figure 7.1 depicts a bijection.

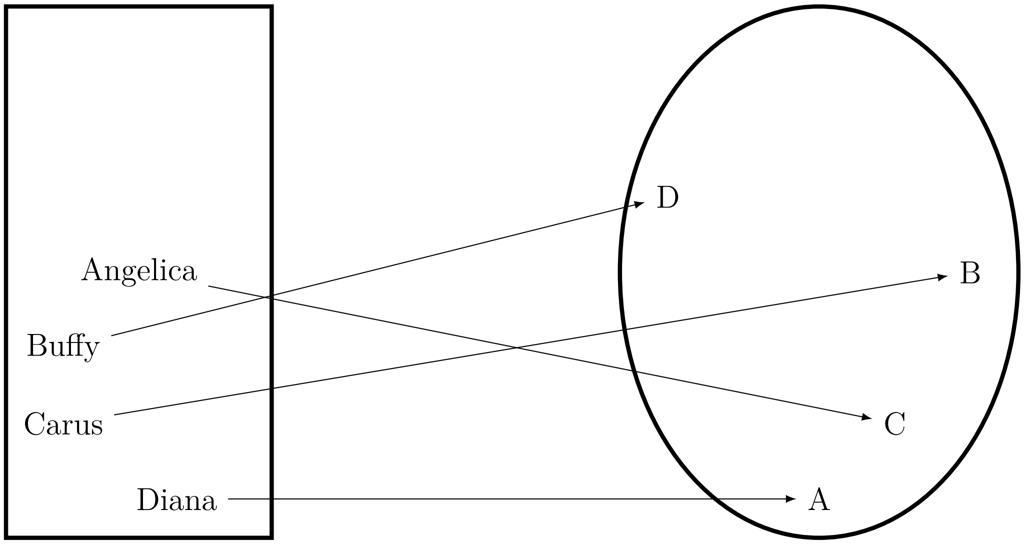

Figure 7.2:

7.6 Picture Two

Exercise 7.6.1. Figure 7.2 depicts a function.

Exercise 7.6.2. Figure 7.2 depicts an injection.

Exercise 7.6.3. Figure 7.2 depicts a bijection.

Exercise 7.6.4. If Figure 7.2 depicts a function, and if we call this function f, then $f(\text{Angelica}) = C$.

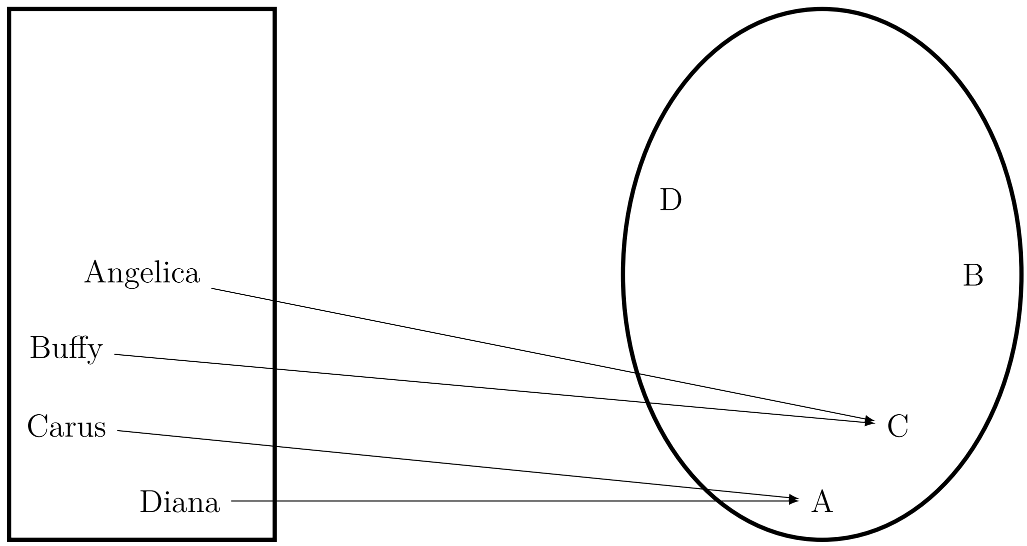

Figure 7.3:

7.7 Picture Three

Exercise 7.7.1. Figure 7.3 depicts a function.

Exercise 7.7.2. Figure 7.3 depicts an injection.

Exercise 7.7.3. Figure 7.3 depicts a bijection.

Exercise 7.7.4. If Figure 7.3 depicts a function, and if we call this function f, then $f(\text{Angelica}) = C$.

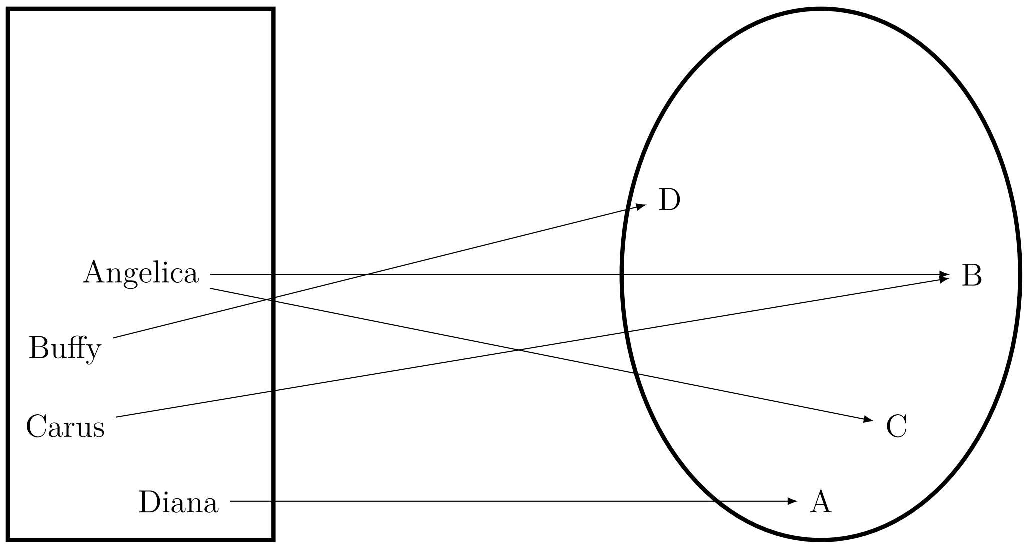

Figure 7.4:

7.8 Picture Four

Exercise 7.8.1. Figure 7.4 depicts a function.

Exercise 7.8.2. Figure 7.4 depicts an injection.

Exercise 7.8.3. Figure 7.4 depicts a bijection.

7.9 Challenge problems

Exercise 7.9.1. Show that if there exists a bijection from X to Y, then there exists a bijection from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$.

Exercise 7.9.2. Exhibit an injection from X to $\mathcal{P}(X)$.

Exercise 7.9.3. If $A \subset B$, prove that $\mathcal{P}(A) \subset \mathcal{P}(B)$.

Exercise 7.9.4. Exhibit an injection from $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$.