Lecture 8

Set notations, unions, intersections

8.1 Honest advice for the rest of the semester

We've gotten some familiarity with the basics of careful thinking and precise language. If you haven't scored well on your writing assignments (consistently scoring 7/10 or below), the honest advice is that you will need to catch up, and you need to seek extra help during office hours or by making appointments. All assignments hereon depend on precise writing.

We've also gotten some familiarity with the basics of sets, subsets, functions, and elements. If you scored below an 85 percent on the exam, the honest advice is that you will need to catch up, and you need to go back and make sure you understand how/what you got incorrect.

Everything from now on will build on the above skill sets.

That was some real talk. Math is fun and creative, but getting a degree involves being assessed for your proficiency in necessary math skills. Because this course is meant to prepare you for other courses in the math major, we will be ramping up.

If the above words were a bit disheartening or intimidating, that is okay. Name that fear, name that discouragement; now let's name the thing that will get you to succeed in this class: Asking the right questions, not being afraid of being wrong, and challenging yourself to make sure you are understanding the material.

8.2 Two new notations

Before we go on, let me introduce two new blackboard-bold letters.

Notation 8.2.1. $\mathbb{Q}$ represents the set of rational numbers.
$\mathbb{Z}$ represents the set of all integers.

Example 8.2.2. $\frac{23}{\in}\mathbb{Q}$, and $-\frac{3}{19} \in \mathbb{Q}$, and $\frac{-5}{20} \in \mathbb{Q}$.
Likewise, we have $-3, -2, -1, 0, 1, 2, 3 \in \mathbb{Z}$.

Remark 8.2.3. The symbol $\mathbb{Z}$ is blackboard bold font for the capital letter Z. Why do we use Z for the integers? In German, the word for number is Zahl (with plural form Zahlen).

Remark 8.2.4. The symbol $\mathbb{Q}$ is blackboard bold font for the capital letter Q. The Q stands for "quotient." Remember from homework that a rational number is a number that can be expressed as a fraction of two integers, and fractions are quotients.

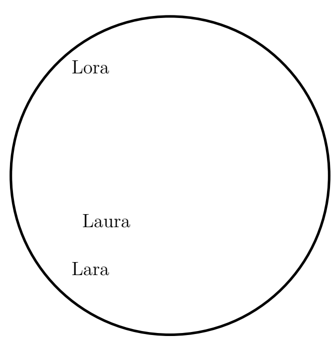
We also saw from homework the comforting fact that all rational numbers have a decimal expansion that eventually repeat some string of decimals (and vice versa).

8.3 Set notation

We have mostly "drawn" sets, or described sets using long sentences. There is a notation that mathematicians used to describe sets, called set notation. Let's begin by example.

Example 8.3.1. The notation {Lara, Laura, Lora} represents a set containing exactly three elements, and the elements are Lara, Laura, and Lora.

Before this lecture, we could have drawn the set {Lara, Laura, Lora} as follows:



The main things to take away:

- The curly brackets $\{\}$ signal that we are speaking of a set, in the same way that the outline circles in drawings tell us we are speaking of sets.
- The text between/inside the curly brackets tell you the elements of the set.

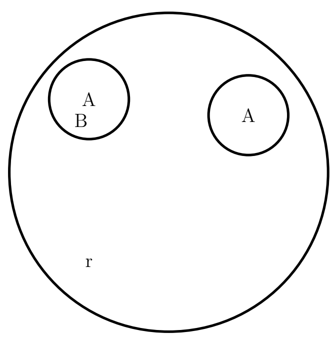
Example 8.3.2. $\{1, 3, 5\}$ is the set containing exactly three elements, called 1, 3, and 5.

Example 8.3.3. $\{\}$ is the empty set.

Using this notation, you can also describe sets of sets:

Example 8.3.4. $\{\{A, B\}, r, \{A\}\}$ is a set containing exactly three elements. One of these elements is called r. The other two elements are themselves sets—a set containing exactly A and B, and a set containing only A.

A picture of $\{\{A, B\}, r, \{A\}\}$ is as follows:



Example 8.3.5. Let $X = \{a, b\}$. This means that X is a set containing exactly two elements called a and b.

Let $\mathcal{P}(X)$ denote the power set. Then we can write:

$$\mathcal{P}(X) = \{\{\}, \{a\}, \{b\}, \{a, b\}\}.$$

If you know what the power set is, there is no new knowledge being conveyed in the equation. The above equation is merely saying that the power set of X is a set containing four elements: The empty set, the set containing only a, the set containing only b, and the set containing exactly two elements: a and b.

Note that the above equality has content! The lefthand side is defined as "the set of all subsets of X." The righthand side has the content of explicitly writing out all the elements of $\mathcal{P}(X)$.

Warning 8.3.6 (Order doesn't matter)**.** The fact that we read left to right can introduce some biases in our minds. Consider the sets

$$A = \{a, 1, I\}, \qquad B = \{a, I, 1\}, \qquad C = \{1, a, I\}.$$

The way that the elements of these three sets are listed may make you believe that there is some preferred ordering of the elements. But there isn't. In fact, A, B, and C are all the same sets!

This is one reason I wanted us to begin by drawing sets as blobs with random things in them. In a blob, there is *no* order imposed on elements.

But when we write sets using curly bracket notation, we have to choose an order (from left to right) in which to write the elements. This introduces bias into our thinking; but to a set, there is *no* order to its elements. For example, the sets

$$\{1, 3, 5, 7\} \qquad \{1, 5, 3, 7\} \qquad \{7, 3, 5, 1\} \qquad \{1, 3, 7, 5\}$$

are all the same set.

8.4 Set notation using defining properties (set-builder notation)

So far we've only seen examples of set notation where our sets are finite; even more, we've seen relatively small examples where we can explicitly write out the elements of a set onto a sheet of paper.

But what if you want to talk about the set of all even natural numbers? Well, you're probably familiar with the "dot dot dot" notation:

Example 8.4.1. The notation $\{0, 2, 4, 6, 8, 10, 12, \ldots\}$ can be used to refer to the set of all even natural numbers.

The "dot dot dot," i.e., the "$\ldots$" means "and so on" or "and so forth." It can be a useful notation if the writer and the reader have a strong mutual understtnding of how the list of numbers should continue.

Warning 8.4.2 (Beware the ... notation)**.** Because you, as a reader, probably know that the above is the beginning of a list of even numbers, you are probably fine with this ... notation.

However, if somebody were to define a set by writing

$$F = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 45, 79, 124, \ldots\}$$

you may be frustrated, because it may not be obvious how to continue the pattern.

So, be warned: There is something imprecise and informal about the ... notation. If the pattern is not clear to the reader, you are not being precise enough.

We will heed this warning. In math, it is *extremely* common to instead define things by saying what *type* of things we are considering, and what *condition(s)* we demand of them. For example:

Example 8.4.3. Consider the set

$$\{a \in \mathbb{N} \text{ such that } a \text{ is even}\}.$$

This would be read out loud as follows: "The set of those a in $\mathbb{N}$ such that a is even."

In other words, the above notation represents the set of even natural numbers. The inside of the brackets tell us that we are considering a set containing natural numbers (the type of stuff in the set), but only the even ones (the condition we put on the stuff in the set).

Definition 8.4.4. The use of curly brackets, combined with the format of "type" and "condition" is often called *set-builder notation*.

Notation 8.4.5. In fact, mathematicians are so lazy that they will often write the abbreviation "s.t." to mean "such that," and they will often write a single vertical bar | to mean "such that," or a colon symbol : to mean the same thing.

Example 8.4.6. All the following expressions mean the exact same thing:

$$\{a \in \mathbb{N} \text{ such that } a \text{ is even}\}.$$

$$\{a \in \mathbb{N} \text{ s.t. } a \text{ is even}\}.$$

$$\{a \in \mathbb{N} \,|\, a \text{ is even}\}.$$

$$\{a \in \mathbb{N} \,:\, a \text{ is even}\}.$$

They would all be read out loud as follows: "The set of those a in $\mathbb{N}$ such that a is even."

Example 8.4.7. Here are some more examples of set-builder notation:

(a) Fix a set X. Then $\{A \,|\, A \subset X\}$ is the power set of X. (In this example, there is no commentary on what "type" of thing the A in this set are supposed to be; it is only from the condition—$A \subset X$—that we realize A must be some set.

(b) $\{x \in \mathbb{R} \,|$ There exist integers a and b for which $x = a/b\}$ is the set of rational numbers.

(c) $\{n \in \mathbb{N} \,|$ There exists a natural number x so that $x^2 = n\}$ is the set of all square numbers.

Example 8.4.8. Here is a non-mathematical example, in case it helps. Let E be the set of all elephants, and let $P = \{e \in E \,|\, e \text{ is purple}\}$. Then P is the set of all purple elephants.

8.5 Unions

There are many ways to make new sets out of old sets. We'll first begin with the idea of taking unions of sets.

8.5.1 Union of two sets

Let A and B be two sets. Then the *union of A and B* is denoted

$$A \bigcup B$$

and is defined to be the following set:

$$\{x \,|\, x \in A \text{ or } x \in B\}.$$

In other words, the union of A and B is the collection of elements that are either in A, or in B (or possibly both!).

You should think of $\bigcup$ as a giant U. However, it is customary that we never write a "tail" on this U. For example, we would never write $A\mathcal{U}B$. When writing the symbol, you should really think of the union symbol $\bigcup$ as a horseshoe shape, or like a rotated subset symbol, rather than as the letter U.

Example 8.5.1. (a) $\{1,2,3\} \bigcup \{4,5,6\} = \{1,2,3,4,5,6\}$.

(b) $\{1,2,3\} \bigcup \{3,4,5\} = \{1,2,3,4,5\}$. Note that the union does not "double count" or "create duplicates" of elements that are in both sets. (You notice this by paying attention to the element 3 in this example.)

(c) For any set A, we have that $A \bigcup A = A$.

(d) For any set A, we have that $A \bigcup \emptyset = A$.

Warning 8.5.2. It's hard to think of a physical analogy for union of sets. The "bags" analogy becomes subtle for the following reason: If you have two bags called A and B, to "combine" the two bags, you'd dump the contents of both bags into a third bag.

However, if A and B happen to be bags that share some comment element (for example, if both contain an element called John), this dumping imagery becomes wonky because it feel like the third bag should then contain two copies of that element (two Johns).

8.5.2 Union of a collection of sets

Now, suppose somebody gives you a lot of sets—for example, five sets called A, B, C, D, E. Then you can guess what I mean by the union of these five sets:

$$A \bigcup B \bigcup C \bigcup D \bigcup E = \{x \mid x \in A \text{ or } x \in B \text{ or } x \ inC \text{ or } x \in D \text{ or } x \in E\}$$

or, put more simply, the union is the set of all elements that are elements in at least one of A, B, C, D, E.

But what if we took 27 sets? Conceptually, you probably know exactly what the union of those 27 sets is—the set consisting exactly of those elements appearing in at least one of those 27 sets.

But how would you write this out? It would take too long, and we want to be a bit lazy about it.

So let's first talk about how to write a collection of sets (for example, a collection of 27 sets).

Example 8.5.3. It is very common to write something like: Consider a collection of sets

$$U_1, U_2, U_3, \ldots, U_{27}.$$

Now, somebody needs to tell you what U_1 is, what U_2 is, what U_3 is, and so forth up to U_{27}. But you can see at least that the above list of sets saves you the headache of writing all 27 sets out.

So, if that's how you write a list of sets, how would you write their union? One way you could write it is

$$U_1 \bigcup U_2 \bigcup U_3 \bigcup \ldots \bigcup U_{27}.$$

But a way that takes up even less space is:

$$\bigcup_{i=1}^{27} U_i.$$

What this notation means is: Take the union of the sets U_i, where i takes values between 1 and 27. If you've taken Calculus I or Calculus II, this is probably reminiscent of summation notation.

But hold on. What if you want to take a union of *infinitely* many sets? (You'll do this for your homework.) This doesn't mean each set U_i is infinite; rather, I mean what if you want infinitely many values of i?

Again, we'll want to think about a way to write down the infinite *collection* of sets to begin with.

Warning 8.5.4. I strongly *discourage* you from writing a collection of sets as

$$U_1, U_2, U_3, \ldots.$$

As we will see later in this course, there are infinite collections that *cannot* be enumerated by positive integers as above. It will do a great disservice if you believe every infinite collection of things can be listed using dot-dot-dot notation.

Example 8.5.5. For every real number $t \in \mathbb{R}$, you can define an open interval $U_t = (t-1, t+1)$.

I want us to consider that there is one set U_t for every $t \in \mathbb{R}$. In other words, we have an infinite collection of sets; some of these sets look like U_π, U_1, U_0, $U_{\sqrt{2}}$, U_{-3}, et cetera.

It would not make sense to list this collection using a 1, 2, 3, ... method. In fact, we'll see later in the course that it's impossible to do that for this example.

Example 8.5.6. For no good reason, you might also define a set as follows. For every rational number $x = a/b \in \mathbb{Q}$, define A_x to be the open interval $(x-3, x+3) \subset \mathbb{R}$.

Then we have a collection of sets—a set A_x for every rational number x.

As it turns out, in most examples in life, a collection of sets is *indexed* by another set.

Definition 8.5.7 (Indexing set)**.** This "another set" is called the *indexing set* of the collection.

Example 8.5.8. In Example 8.5.6, the indexing set is $\mathbb{Q}$. In the example before that, the indexing set is $\mathbb{R}$. In Example 8.5.3, the indexing set is the set $\{1, 2, \ldots, 27\}$. The idea is that for every element of the indexing set, you associate a set.

Now we know how to *write* the union of a collection of sets indexed by some indexing set $\mathcal{A}$.

Notation 8.5.9. In general, suppose you have an indexing set $\mathcal{A}$, and that for every $\alpha \in \mathcal{A}$, you can specify a set U_α. We let

$$\{U_\alpha\}_{\alpha \in \mathcal{A}}$$

denote the collection of sets. It is a set of sets. For example, for every $\alpha \in \mathcal{A}$, there is an element called U_α in the above set. If you chose another element $\alpha' \in \mathcal{A}$, there is a corresponding element $U_{\alpha'}$ in the above collection.

Definition 8.5.10 (Union of a collection of sets)**.** Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of sets. The the union of this collection is defined to be the set

$$\{x \,|\, \text{there is some } \alpha \in \mathcal{A} \text{ for which } x \in U_\alpha\}.$$

In other words, the union of this collection is the set of all elements that appears in at least one of the U_α.

Finally, we write

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha$$

to represent the union of this collection.

Remark 8.5.11 (The most important remark)**.** The idea of the "infinite" can be intimidating. But the definition of union actually tells you that you can think about each U_α one at a time.

More precisely, suppose you wonder whether a given x is an element of the union $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$.

Then all you need to is find out whether there is (at least) *one* α for which $x \in U_\alpha$.

8.6 Intersections

Now we'll talk about another way to construct new sets out of old: Taking intersections.

8.6.1 Intersection of two sets

Definition 8.6.1. Let A and B be two sets. The *intersection of A and B* is the set of all elements that are in both A and B. In other words, the intersection of A and B is the set

$$\{x \,|\, x \in A \text{ and } x \in B\}.$$

Sometimes, we will also say the intersection of A *with* B as well.

Example 8.6.2. Here are a few examples:

(a) If $A = \{1, 2, 3\}$ and $B = \{2, 3, 5, 6\}$ then the intersection of A and B is $\{2, 3\}$.

(b) If A is any set, then the intersection of A with the empty set is the empty set.

(c) If A is any set, then the intersection of A with itself is A.

(d) If $A \subset B$, then the intersection of A and B is A.

Notation 8.6.3. The intersection of A and B will be denoted by $A \bigcap B$.

Example 8.6.4. We can thus re-write the above examples as follows:

(a) $\{1,2,3\} \bigcap \{2,3,5,6\} = \{2,3\}$.

(b) $A \bigcap \emptyset = \emptyset$

(c) $A \bigcap A = A$.

(d) If $A \subset B$, then $A \bigcap B = A$.

Remark 8.6.5. Let's compare two particular definitions of union and intersection. We have:

$$A \bigcup B = \{x \,|\, x \in A \text{ or } x \in B\}.$$

$$A \bigcap B = \{x \,|\, x \in A \text{ and } x \in B\}.$$

The only difference in the notions of intersection and unions is the word or/and. A lot of professors like to emphasize that "union" is a set-theoretic way of expressing the idea of "or," and that "intersection" is a set-theoretic way of expressing the idea of "and."

8.6.2 Intersection of a collection of sets

Just as with unions, we will sometimes want to consider the intersection of many sets as once—not just two.

So as before, suppose we have a collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of sets.

Definition 8.6.6. The intersection of the sets in the collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$, sometimes abbreviated as "the intersection of the U_α", is the set

$$\{x \,|\, \text{for every } \alpha \in \mathcal{A},\, x \in U_\alpha.$$

We use the following notation to denote this intersectino:

$$\bigcap_{\alpha \in \mathcal{A}} U_\alpha.$$

In other words, x is in the intersection of the collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ when x is an element of *every* U_α.

Example 8.6.7. Let $\mathcal{A}$ be the set of all positive integers (so $\mathcal{A}$ has elements 1, 2, 3, 4, etc.), and for every $n \in \mathcal{A}$, define the set U_n to be the set of all integers that are multiples of n. So for example, U_1 is the set of all integers, while U_2 is the set of all even integers and U_3 is the set of all integers divisible by 3, and so forth.

Then the intersection $\bigcap_{n\in\mathcal{A}} U_n$ is the set consisting of exactly one element called zero. Put another way, zero is the only number that is a multiple of all positive integers.

Remark 8.6.8. How do you check whether an element x is in the intersection $\bigcap_{\alpha\in\mathcal{A}} U_\alpha$? Looking back at the definition, it suffices to answer a question about x *one α at a time.* This is similar to how we have to think about unions of large collections of sets—even if we have many sets, all our verifications will come down to one question at a time.

So to prove that x is an element of $\bigcap_{\alpha\in\mathcal{A}} U_\alpha$, you just to show that: If you choose an α, and regardless of which α you choose, x is an element of U_α.

Remark 8.6.9. Before, we saw that the difference between union and intersection is the difference between "and" and "or."

Well, this is still true, but let's look at the cleanest way we wrote our definitions:

$$\bigcup_{\alpha\in\mathcal{A}} U_\alpha = \{x \,|\, \text{for some } \alpha \in \mathcal{A},\ x \in U_\alpha.\}$$

$$\bigcap_{\alpha\in\mathcal{A}} U_\alpha = \{x \,|\, \text{for every } \alpha \in \mathcal{A},\ x \in U_\alpha.\}$$

In other words, another way to think about union and intersection is as follows. Unions are about "is x in *some* U_α"—meaning can we show that there exists at least one α for which $x \in U_\alpha$—while intersections are about "is x is *every* U_α"—meaning can we show that, regardless of which α we choose, $x \in U_\alpha$.

When you come down to wanting to understand whether x is in a union, or in an intersection, you'll thus be flexing two different mental muscles.

The "union" muscle will test whether you can produce an example of an α that plays well with x. ("Plays well" means $x \in U_\alpha$ here; I'm using an informal, non-mathematical term.)

The "intersection" muscle will test whether you have an abstract understanding of what it means for α to be an element of $\mathcal{A}$; this abstract understanding will allow you to say whether x is thus in U_α regardless of the choice of α.

Of course, in both cases, you have to understand how α defines U_α—this will be given to you in the problem at hand, and the mental muscle that this works will be different in every context. Some sets have very crazy definitions.

8.7 Exercises

Exercise 8.7.1. Let $A = \{n \in \mathbb{N} \,|\, n \text{ is a prime number and } 3 \leq n \leq 8\}$. Which of the following is equal to A?

(a) $\{5, 3, 7\}$

(b) $\{7, 5, 3\}$

(c) $\{n \in \mathbb{N} \,|\, n \text{ is a prime number and } 3 \leq n \leq 13\}$

(d) $\{n \in \mathbb{N} \,|\, n \text{ is a prime number and } 3 \leq n \leq 10\}$

(e) $\{n \in \mathbb{N} \,|\, n \text{ is an odd number and } 3 \leq n \leq 8\}$

(f) $\{3, 5, 7\}$.

Exercise 8.7.2. Which of the following is equal to the set of rational numbers?

(a) $\mathbb{R}$

(b) $\{x \in \mathbb{R} \,|\, \text{ there exists two integers } a, b \text{ for which } x = a/b\}$.

(c) $\{x \in \mathbb{R} \,|\, \text{ the decimal expansion of } x \text{ eventually repeats a string of digits}\}$.

(d) $\{y \in \mathbb{R} : \ y \text{ is not irrational}\}$.

(e) $\{a \in \mathbb{R} : \text{the decimal expansion of } a \text{ eventually ends with 0 repeating}\}$.

Exercise 8.7.3. Which of the following is a true statement? Select all that are true.

(a) $\{5, \pi, e\} \bigcup \{5, \pi, 1\} = \{1, e, \pi, 5\}$.

(b) $\{5, \pi, e\} \bigcup \{5, \pi, e\} = \{e, \pi, 5\}$.

(c) $\{5, \pi, e\} \bigcup \{5, 3\} = \{5, 3\}$

(d) $\{5, \pi, e\} \bigcap \{5, 3\} = \{5, 3\}$

(e) $\{5, \pi, e\} \bigcap \{5, \pi, 1\} = \{\pi, 5\}$.

(f) $\{5, \pi, e\} \bigcap \{5, \pi, e\} = \{e, \pi, 5\}$.

Exercise 8.7.4. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of sets. Which of the following is a true statement? Select all that are true.

(a) $\mathcal{A}$ has infinitely many elements.

(b) $\mathcal{A}$ is a finite set.

(c) $\mathcal{A}$ is called the index set, or the indexing set, of the collection of sets.

(d) For every $\alpha \in \mathcal{A}$, U_α is a set.

Exercise 8.7.5. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of sets. Which of the following is a true statement? Select all that are true.

(a) x is an element of $\bigcap_{\alpha \in \mathcal{A}} U_\alpha$ if and only if: For every $\alpha \in \mathcal{A}$, x is an element of U_α.

(b) x is an element of $\bigcap_{\alpha \in \mathcal{A}} U_\alpha$ if and only if: For some $\alpha \in \mathcal{A}$, x is an element of U_α.

(c) x is an element of $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$ if and only if: For every $\alpha \in \mathcal{A}$, x is an element of U_α.

(d) x is an element of $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$ if and only if: For some $\alpha \in \mathcal{A}$, x is an element of U_α.

Exercise 8.7.6. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of sets. Which of the following is a true statement? Select all that are true.

(a) It is possible that $\bigcap_{\alpha\in\mathcal{A}} U_\alpha$ equals $\emptyset$.

(b) It is possible that $\bigcup_{\alpha\in\mathcal{A}} U_\alpha$ equals $\emptyset$.

(c) It is possible that $\bigcup_{\alpha\in\mathcal{A}} U_\alpha = \bigcap_{\alpha\in\mathcal{A}} U_\alpha$.

(d) It is impossible that $\bigcup_{\alpha\in\mathcal{A}} U_\alpha = \bigcap_{\alpha\in\mathcal{A}} U_\alpha = \emptyset$.

(e) For every $\alpha \in \mathcal{A}$, $U_\alpha \subset \bigcap_{\alpha\in\mathcal{A}} U_\alpha$.

(f) For every $\alpha \in \mathcal{A}$, $U_\alpha \subset \bigcup_{\alpha\in\mathcal{A}} U_\alpha$.