Lecture 9

Axiomatic thinking and Peano's axioms

Your math education so far has probably been about learning what other people have discovered. And it is very, very rare for you to learn this stuff *while being told what those discoverers knew.*

For example, did you know that the first known European person to try to understand imaginary numbers was also the first European person to systematically start using negative numbers? In other words, did you know that both imaginary and negative numbers actually came into European civilization's consciousness at around the same time? (For more, you can learn about the Italian mathematician Girolamo Cardano.)

You might have never expected that. You knew about negative numbers in elementary school. You didn't learn about imaginary numbers until high school or college!

The weird thing about math education is that it's like playing a video game, but from a save point that someone else got you to. You didn't have to invent numbers—someone got you to that save point, and you just needed to keep playing. You didn't have to discover the rules of taking derivatives—someone got you to that save point, and you took it from there.

9.1 Level zero: Axioms

So, let's assume that you're beginning the video game of math from level zero. You turn on the switch, or boot up the app, or however else you want to think of it.

What "moves" are you allowed to use? Where are you beginning? What *is* level zero? In other words, what mathematical facts are we even allowed to employ to get to another level of the game?

You probably expect this to have an answer. Here's the truth: There isn't one.

In fact, to begin playing with *any* system of logic, you have to *choose* what level zero is. In other words, you just declare that some sentences have to be true. You don't even define what the words in those sentences are (you can't, because to define them, you'd have to in turn know what the words you use in your definitions mean! So then what do those definitional words mean? Were they explained in level -1?). You just declare that some sentences, with some words in them, are true.

The above paragraph gives the least romantic description of an *axiom*.

An *axiom* is a truth that you declare to be true at the beginning of your video game. You declaring your axioms is like the Big Bang for your logical universe.

And the point is, there is a video game for every collection of axioms you choose. In other words, the logical conclusions you can reach can be *different* if you begin with a different set of axioms!

Example 9.1.1. For example, Daniela's axioms could simply be the following sentences:

1. All bananas are delicious.
2. Every apple is a banana.

It does not matter whether your axioms conflict with somebody else's. (We all know apples are not bananas, but remember—what *we* think these words mean don't matter. Every collection of axioms must consist of undefined words, and here, "banana" may as well be the word "Planet Omicron 6.")

Based on these axioms, Daniela could conclude: Every apple is delicious. So Level Zero began with Daniela's axioms. Concluding that every apple is delicious gave us another new fact of the universe, and got us to Level One.

Example 9.1.2. An equivalent collection of axioms is:

1. Every burple is deftastic.

2. Every snarf is a burple.

Then you could conclude that every snarf is deftastic. Note that "burple, deftastic, snart" have no meanings. But regardless of what they mean, the logical conclusion is that every snarf is deftastic.

Example 9.1.3. It is possible to choose a completely broken collection of axioms. For example:

1. Every burple is deftastic.

2. Every snarf is a burple.

3. There exists at least one snarf that is not deftastic.

The third axiom directly contradicts a logical conclusion of the first two axioms. After all, if every snarf is deftastic, how could there be a snarf that is not deftastic? But, based on the axioms, the statement that "there exists at least one snart that is not deftastic" is both provably true, and provably false.

Such axioms are called *inconsistent.* If you begin with an inconsistent collection of axioms, it turns out you can prove that any statement is true, and that any statement is false. This sounds interesting—to know that inconsistent axioms exist—but that's about as far as this goes. You'd never want to work with an inconsistent collection of axioms. They're all equivalent, and they're all useless.

Remark 9.1.4. So, in fact, there is an entire field of math that wonders what our axioms *ought* to be. There are many commonly accepted axioms, and for now, it seems the most widely accepted axioms are called the ZF axioms, after Ernesto Zermelo and Abraham Fraenkel (both German mathematicians, though the latter moved to Israel for most of his later life). These axioms were developed by these two mathematicians (and others) and probably came into final form sometime in the 1920's.

Remark 9.1.5. And in fact, very smart mathematicians *have* made mistakes with Level Zero. There are indeed ways to feel happy with set theory and yet construct contradictions! For example, it turns out that the logical universe is inconsistent if you allow the existence of a set of all sets. We might see that later in class. For now, you can look up Russell's Paradox.

9.2 Our level zero for the natural numbers

So there are many level zeros. And for now, I would like to focus on how we know that the *natural numbers* exist.

Once upon a time, I made a big fuss about the possibility of an alien species not knowing what we mean by "next." This is where the discussion begins.

First, even though we may not "know" that natural numbers exist from Level Zero, we know in our hearts that they exist. I'd like to write down some properties that we know about natural numbers:

(N1) First, there's at least one natural number.

(N2) Second, there's always a "next" natural number. More precisely, given a natural number n, there's a natural number called +1. This "next" number is unique, in that if $n + 1 = m + 1$, then we know $n = m$.

(N3) The only natural number that's not a "next" natural number is zero.

(N4) Finally, every natural number is obtain from zero by applying +1 some number of times.

Okay, great. I think if we read over that list carefully, we would all accept that the above are all true statements about $\mathbb{N}$. However, there is a very big flaw in the last statement: "applying +1 some number of times." The notion of "number" we use there is circular! What does it mean to add +1 some number of times? Do we know what we mean by that if we do not know what a natural number is?

This is both vexing and fascinating. What even is a number?

One of the first solutions to this issue was developed by Giuseppe Peano (Italian, 18580-1932). He very cleverly translated all of the above into a statement about *sets*. He noticed the following about the set of natural numbers:

(P$\mathbb{N}$1) $\mathbb{N}$ is a non-empty set.

(P$\mathbb{N}$2) There is a function from $\mathbb{N}$ to $\mathbb{N}$, we'll call it t, which is injective.

(P$\mathbb{N}$3) Moreover, there is a unique element of $\mathbb{N}$ that is not hit by t. We will call this unique element z.

(P$\mathbb{N}$4) Finally, suppose that S is some subset of $\mathbb{N}$ for which (i) $z \in S$, and (ii) Whenever $n \in \mathbb{N}$, then $t(n) \in \mathbb{N}$. Then $S = \mathbb{N}$.

You should verify that each of these statements is equivalent to the first facts about natural numbers we just talked about. The translation is by setting $z = 0$ and t to be the function sending n to $n + 1$.

The most amazingly clever part of these statements is the last bit: Instead of talking about "applying +1 a number of times," he completely avoids the notion of number by cleverly articulating a characterization of subsets of $\mathbb{N}$. It gives a formal description of the intuition that $\mathbb{N}$ is the "smallest" thing you can produce out of 0 and out of +1.

Now, here is a very common thing that mathematicians do now. This is your first time seeing it, so it seems like cheating. It's not. We *declare* that there must exist a set and a function satisfying the properties above—in other words, we make the existence an axiom. The axiom(s) asserting the existence of $\mathbb{N}$ and t are called Peano's axioms.

9.3 Peano's axioms

Here is one formulation of Peano's axioms. For it, we assume that we know what we mean by "sets" and "functions" and "injection."

Peano's axioms: There exist a set N, and a function t from N to N, satisfying the following properties:

(PA1) N is a non-empty set.

(PA2) t is an injection.

(PA3) There is a unique element of N that is not hit by t. We will call this unique element z.

(PA4) Finally, suppose that S is some subset of N for which (i) $z \in S$, and (ii) Whenever $n \in N$, then $t(n) \in N$. Then $S = N$.

If we assume Peano's axioms, we know that there exists a set N equipped with a function t of it, and a unique element $z \in N$ with some nice property. In other words, we are *not proving* the existence of N and t; we are just taking it for granted as level zero.

We then *define the following notation*:

Notation 9.3.1. Suppose N is a set, and t a function, guaranteed by Peano's axioms. Then we use the notation $\mathbb{N}$ to denote N.

We let 0 denote the element $z \in N$.

We let 1 denote the element $t(0)$.

We let 2 denote the element $t(1)$.

More generally, given an element $n \in N$, we let $n+1$ denote $t(n)$.

Thus, we are defining "+1" in terms of t. This is an *incredibly* abstract way of thinking about the natural numbers. It is no longer something that measures size; it is merely characterized as a set with some notion of "next" as made precise by t.

Remark 9.3.2. If we have time in this course, we will see another set of axioms—an even more "beginner's" Level Zero, that will imply the existence of a set N with a function t as above. In other words, though Peano took the above as axioms, it turns out we can create a "deeper" set of axioms that imply the existence of the things Peano wanted.

Remark 9.3.3. This is how abstract mathematics actually goes now-a-days. We realize that there are some properties that we take for granted, or use, all the time. It is the properties we use that are useful, so we name the properties we use. In doing so, we hope to give a *new* definition or characterization of the thing we are studying: It used to have a familiar meaning, but we strip it to have only the properties we care about. Peano's axioms are one of the first examples of this principle in action.

Let me emphasize again that this may feel like cheating. That's okay. But there is also brilliance. It is incredibly hard to stare at a familiar thing and realize it is nothing more than a few properties. The deepest mathematical insights are of this nature.

9.4 The inductive axiom

As I've mentioned, by far, the most clever part of Peano's axioms is (PA4). (Finally, suppose that S is some subset of N for which (i) $z \in S$, and (ii) Whenever $n \in N$, then $t(n) \in N$. Then $S = N$.)

(PA4) is called the *inductive* axiom.

It is called the inductive axiom because it states that the set of all natural numbers is "induced" by beginning with 0, and then just adding 1 repeatedly.

9.5 Application: Induction

And using the inductive axiom (or, if you prefer, the inductive property of the natural numbers) we can prove ridiculously powerful theorems.

We will see more of this next time, when we learn about proofs by induction.