Lecture 10

Induction

Last time, we saw **Peano's axioms:** There exist a set N, and a function t from N to N, satisfying the following properties:

(PA1) N is a non-empty set.

(PA2) t is an injection.

(PA3) There is a unique element of N that is not hit by t. We will call this unique element z.

(PA4) Finally, suppose that S is some subset of N for which (i) $z \in S$, and (ii) Whenever $n \in N$, then $t(n) \in N$. Then $S = N$.

As I mentioned last time, the very clever part of these axioms are contained in (PA4): Without saying "a natural number is obtained from 0 by adding 1 some number of times," Peano managed to abstractly encapsulate this idea (without ever using the notion of number, or of adding 1!).

Remark 10.0.1. But here's an unsettling part of the axioms. Okay, so there exists a set N and a function t satisfying all the above properties. What if there are *many* such sets? In our intuition, isn't there only "one" set of natural numbers?

We will answer this question next time. But think on it if you have the time.

Remark 10.0.2. I want to remark that you are learning two things at once here: One is cultural, in that you are learning about a way that somebody in the past came up with axioms to guarantee the existence of the set of natural numbers.

The other is mathematical. (PA4) should somehow be "intuitively true" if you understand that the axiom is simply saying: Begin with zero, add 1 to zero a bunch of times; you'll eventually get any natural number you want. The mathematics is that this is codified using only the language of sets and functions.

Moreover, this property is *the* property that determines what the set of natural numbers is. There is always a "next," and the numbers obtained by going "next" to 0, then next to that, then next to that, and so forth, are the natural numbers.

Axiom (PA4), which you should think of as the most important property of the natural numbers, is called the **em inductive axiom.** It is because it says that the natural numbers are "induced" (that's not a mathematically precise term; it's more poetry) by beginning with 0 and adding 1 many times.

Today, what we'll do is talk about a *consequence* of this property of the natural numbers. It is called proof by induction.

10.1 Statements that can be tested on all natural numbers

Here are some mathematical claims:

1. If a function f is the derivative of F, then $\int_a^b f\,dx = F(b) - F(a)$.

2. The sum of all integers between 0 and n is equal to the number $n(n+1)/2$.

3. A collection of n objects admits exactly $n!$ orderings. (That is, if you have n objects and you want to number them 1 through n, there are exactly $n! = n \times (n-1) \times \ldots \times 2 \times 1$ ways of doing this.)

4. If n is any natural number, then $n^3 + 2n$ is divisible by 3.

These all seem like different *kinds* of statements. The first (the fundamental theorem of calculus) is about all functions f having an antiderivative F, and is mainly a computational tool (for computing integrals, for example).

The next two are also for computation—they tell you how to produce a number you want. The last statement isn't so much about computation, but about understanding a property of a funny-looking expression. In either case, the last statement shouldn't seem so obvious at first glance.

But here's something that the last three all have in common: They are statements about natural numbers. More precisely, they assert facts about *all* natural numbers.

Today, we'll learn a proof method called proof by induction, which gives you a very powerful strategy for proving statements about all natural numbers.

Remark 10.1.1. In fact, proof by induction can also help prove the following kinds of statements:

1. If $n \geq 4$, then $n! \geq 2^n$.
2. If $n \geq 1$ and $x > -1$, then $(1+x)^n \geq 1 + nx$.
3. Let f_i be the ith Fibonacci number, where $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, and in general, $f_i = f_{i-2} + f_{i-1}$. Then for all $n \geq 1$, $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$.

In other words, you can also use induction to prove that a statement is true for *all big enough natural numbers*. (In the first example above, if the natural number is ≥ 4.)

The above examples were taken from a textbook by Rosen.

10.2 The theoretical framework for proof by induction

If you want to prove that a statement is true for all natural numbers, here's how you can do it.

- To begin, prove it for the natural number zero.

  Now, let S be the set of all natural numbers for which the statement is true. You've just shown that $0 \in S$.

- Prove that if $n \in S$, then $n + 1 \in S$. (In other words, prove that if the statement is true for some natural number n, then it must also be true for $n + 1$.) This is called the inductive step.

That's it.

Let's think about why this works. The first step proves that $0 \in S$. The second step proves that if $n \in S$, then $n+1 \in S$. But by the inductive axiom, this means that $S = \mathbb{N}$. And we *defined* S to be the set of natural numbers for which the statement we want is true. So $S = \mathbb{N}$ means that the statement must be true for *all* natural numbers, and we are finished.

In the previous section, I also claimed that you can use induction to prove other kinds of statements—for example, statements that only apply for all natural numbers bigger than or equal to 4. In such a situation, you can

- Prove that the statement is true for the natural number 4.

  Now, let S be the set of all natural numbers for which the statement is true.

- Prove that if $n \in S$, then $n+1 \in S$. (In other words, prove that if the statement is true for some natural number n, then it must also be true for $n+1$.) This is again called the inductive step.

Now, you can convince yourself that S is then the set of all natural numbers bigger than or equal to 4. (After all, S is the set of all natural numbers you can obtain by starting with 4, then adding 1 repeatedly.) Therefore—by definition of S—the statement is true for all natural numbers bigger than or equal to 4.

This is quite powerful, as we'll see in a few examples now.

10.3 Examples of proof by induction

In the previous section, I talked about defining a set S to be "the set of all natural numbers for which a statement is true." In writing out proofs by induction, we often omit S. Here's an example.

Exercise 10.3.1. Prove that for every natural number n, the sum

$$\sum_{i=0}^{n} i$$

is equal to $n(n+1)/2$.

Remark 10.3.2. Let's recall what the summation notation means. The $\sum$ is the Greek letter "capital Sigma." (Lower-case sigma is σ.) Σ is the letter that eventually turned into the Roman S, and it stands for "sum."

The summation means, for every whole number between $i = 0$ and n (these are the bounds of the summation) we add up the value i. So for example, if $n = 4$, the summation could be tediously written out as

$$\sum_{i=0}^{4} i = 0 + 1 + 2 + 3 + 4.$$

This number, of course, is 10. The formula $n(n + 1)/2 = 4 \times (4 + 1)/2 = 4 \times 5/2 = 10$ indeed returns 10 as well. So the formula is correct for $n = 1$.

As we've seen already in this class, "proof by example" is not a proof. While it is within our rights as scientists and students to verify that the formula seems to work in certain examples, as mathematicians, it is our job to prove that the formula works for *all* examples. This is where induction comes in.

Proof. So let's prove that the claim is true for all natural numbers by induction.

The first step is to verify for $n = 0$. Well, here, the summation becomes 0. (We add one term at the value $i = 0$.) On the other hand, $n \times (n+1)/2 = 0 \times 1/2 = 0$. So the claim is true for $n = 0$.

Now suppose that the claim is true for some natural number n:

$$\sum_{i=0}^{n} i = n(n + 1)/2. \tag{10.3.0.1}$$

(That this formula is true means that $n \in S$, using our notation S from Section 10.2.) We must prove that the statement holds for $n + 1$; i.e., we must prove that

$$\sum_{i=0}^{n+1} i = (n + 1)(n + 2)/2. \tag{10.3.0.2}$$

We may compute the lefthand side of (10.3.0.2) as follows:

$$\begin{aligned}
\sum_{i=0}^{n+1} i &= \left(\sum_{i=0}^{n} i\right) + (n+1) \\
&= \left(\frac{n(n+1)}{2}\right) + (n+1) \qquad (10.3.0.3) \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\
&= \frac{n(n+1)+2(n+1)}{2} \\
&= \frac{(n+2)(n+1)}{2}.
\end{aligned}$$

Here, we used (10.3.0.1) (the inductive hypothesis) in deducing line (10.3.0.3). The rest is just algebra.

Thus, the above equations tell us

$$\sum_{i=0}^{n+1} i = \frac{(n+2)(n+1)}{2}$$

which is exactly the claim we wanted to prove in (10.3.0.2).

In summary, we have shown that the claim is true for $n = 0$, and shown that if it is true for a natural number n, then it is true for $n+1$. This finishes the proof. □

Remark 10.3.3. A lot of students think proofs by induction are mostly about formulas; they aren't. The real conceptual underpinning of proofs by induction is that: If a set S (of all natural numbers for which a statement is true–e.g., for which a formula holds) contains 0, and if for all $n \in S$, S also contains $n+1$, then $S = \mathbb{N}$.

That's the concept behind induction. The hard work in executing a proof by induction is in translating what it means for n to be an element of S, and how to use that to show that $n+1$ must also be an element of S.

Let's do another example.

Example 10.3.4. Prove that if n is any natural number, then $n^3 + 2n$ is divisible by 3.

Proof. We first verify the claim for $n = 0$. Then $n^3 + 2n = 0^3 + 2 \times 0 = 0 + 0 = 0$. And indeed, 0 is a multiple of 3 (hence divisible by 3).

Now suppose that the claim is true for some natural number n, so that $n^3 + 2n$ is divisible by 3. This means that $n^3 + 2n = 3m$ for some natural number m.

We must now prove that $(n + 1)^3 + 2(n + 1)$ is divisible by 3. Let us re-write this expression as follows:

$$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2 \\ &= (n^3 + 2n) + 3n^2 + 3n + 1 + 2 \\ &= (n^3 + 2n) + 3(n^2 + n + 1). \qquad (10.3.0.4)\end{aligned}$$

(The above is all algebra—nothing new from this course is being used.) Well, let us now use the inductive hypothesis[1] to re-write (10.3.0.4) as

$$3m + 3(n^2 + n + 1)$$

which in turn can be written as

$$3(m + n^2 + n + 1).$$

In other words, we have written the term we began with, $(n+1)^3+2(n+1)$, as "$3 \times$ (some natural number)", because $m+n^2+n+1$ is a natural number (being a sum of a bunch of natural numbers). In other words, $(n+1)^3+2(n+1)$ is divisible by 3. This completes the proof. □

10.4 Some tips

Sometimes, the hardest part of a proof by induction is understand "what statement are we trying to prove," and "what does that statement mean for $n + 1$."

For instance: In the first example, we needed to understand that the formula

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

[1] In other words, let us use the fact that we assume n is a number for which $n^3 + 2n$ is divisible by 3. This means $n^3 + 2n = 3m$ for some natural number m, as we mentioned above.

becomes the formula

$$\sum_{i=0}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

when we are checking it for $n+1$.

Remark 10.4.1. Some students find proofs by induction to be easier when they rephrase it as follows: Check for $n = 0$. then, assume it's true for $n-1$. Now prove it's true for n.

This pushes some of the work to understanding what the statement becomes for the $n-1$ case (instead of the $n+1$ case).

Remark 10.4.2. The assumption that "the statement is true for n" is often called the *inductive hypothesis*. The *proof* that, under the inductive hypothesis, the statement is true for $n+1$, is called the *inductive step.*

Put another way, the hard part of every induction proof is *usually* the inductive step. There are many parts of it that could be difficult, and usually the hard part is understanding what it means to prove a statement for the $n+1$ case.

Remark 10.4.3. In a class like this, a lot of proofs by induction are about formulas. As a result, most proofs by induction are accomplished by understanding the *difference* between the n formula, and the $n+1$ formula.

10.5 Exercises

Exercise 10.5.1. Using induction, prove the statements in Remark 10.1.1.