Lecture 14

The inverse to a bijection

As another example of how to think about, and write, a mathematical argument, today we're going to prove the following.

Proposition 14.0.1. Let X and Y be sets, and suppose $f : X \to Y$ is a bijection. Then there exists a bijection from Y to X.

Remark 14.0.2. Note that the "order" of X and Y are swapped in the last sentence. Informally, if we have a bijection going in one direction, the proposition guarantees the existence of a bijection going in the other direction.

Remark 14.0.3. In a past class, I gave a "visual" and intuitive reason why the proposition ought to be true. If you are given a drawing of a bijection from a set X to a set Y, just reverse the arrows to obtain a bijection from Y to X. We are going to make this rigorous using the precise language we've learned so far.

Below, I'll write out the proof as you might see it in a textbook. However, I will also append footnotes to explain what's going on.

Remark 14.0.4. In future classes, it will be your intellectual duty to realize that many sentences in proofs require footnotes, and that you will need to provide those footnotes for yourself. (And this is not easy, for anybody. I also routinely see math proofs where I have to think for days why one sentence follows from the previous sentence.)

Proof of Proposition 14.0.1. Let $f : X \to Y$ be a bijection. We must produce a bijection from Y to X.[1]

So, given $y \in Y$, define $g(y)$ to be the unique element of X for which $f(g(y)) = y$.[2] (Let us explain why such an element exists, and why it is unique. Note that at least one x for which $f(x) = y$ exists because f is a surjection. Because f is further an injection, there is exactly one[3] such x. This x is what we declare to be $g(y)$.) This defines a function $g : Y \to X$.

We must prove that g is a bijection.

To see that g is an injection, suppose $g(y) = g(y')$. By definition of g, we know that $f(g(y)) = y$, and likewise $f(g(y')) = y'$. On the other hand, because $g(y) = g(y')$, we know that $f(g(y)) = f(g(y'))$.[4] Writing these equalities together, we see

$$y = f(g(y)) = f(g(y')) = y'.$$

Hence $y = y'$. This shows that g is an injection.

To see that g is a surjection, fix any $x \in X$. Define $y = f(x)$. By definition of g, we see that $g(y) = x$.[5] Thus, g is a surjection.

Because g is both a surjection and an injection, it is a bijection. This completes the proof. $\square$

[1] What does it mean to "produce a bijection?" It means, first and foremost, to define a function.

So we must define a function from Y to X. Then, we will verify that the function is a bijection.

Warning: To define a function is not to just give it a name. "Let g be a function" does not define a function; it just tells the reader that the symbol g represents any old function.

[2] Put another way, $g(y)$ is the element of X that f sends to y.

[3] After all, if there are elements x and x' for which $f(x) = y$ and $f(x') = y$, we can conclude that $f(x) = f(x')$. But f is an injection, so x must equal x'.

[4] After all, a function sends one element of the domain to one element of the codomain—a single element of the domain isn't "double duty" assigned to two elements of the codomain.

[5] This is because x satisfies the property that $f(x) = y$; and by definition, $g(y)$ is the element *blah* of X satisfying $f(blah) = y$.

14.1 Inverses

Now, we encounter something quite interesting. The Proposition itself only states that there exists a bijection from Y to X. We have proven that.

However, in the course of the proof, we have discovered something interesting–we can *explicitly* write a bijection $g : Y \to X$ given f. And g feels special. After all, it satisfies the following properties:

For any $y \in Y$,

$$f(g(y)) = y$$

and for any $x \in X$,

$$g(f(x)) = x.$$

This seems useful! So we give it a name.

Definition 14.1.1. Let $f : X \to Y$ be a bijection. Then the *inverse* to f is the function $g : Y \to X$ such that, for all $x \in X$ and $y \in Y$,

$$g(f(x)) = x \qquad \text{and} \qquad f(g(y)) = y.$$

Remark 14.1.2. Note that g is defined to be "the" inverse. You can check that if g' is another function satisfying the above equations, it is necessarily true that $g(y) = g'(y)$ for every $y \in Y$. In other words, g and g' are the same function.

14.2 Exercises

Exercise 14.2.1. Let $f : X \to Y$ be a bijection. Which of the following is true?

(a) X is the same set as Y.

(b) Fix $y \in Y$. Because f is a surjection, there exists at least one $x \in X$ for which $f(x) = y$.

(c) Fix $y \in Y$. Because f is an injection, there exists at most one $x \in X$ for which $f(x) = y$.

(d) Fix $y \in Y$. Because f is a bijection, there exists exactly one $x \in X$ for which $f(x) = y$.

(e) There exists a bijection from Y to X.

Exercise 14.2.2. Let $X = Y = \mathbb{R}$ be the set of all real numbers, and define a function $f : X \to Y$ by $f(x) = x^2$. Which of the following is a true statement?

(a) $f(3) = 9$.

(b) f is a surjection.

(c) f is an injection.

(d) f is a bijection.

(e) There are elements of the codomain that are not in the image of f.

(f) There is exactly one element of the domain that is sent to $0 \in Y$ by f.