Lecture 15

New sets

15.1 What we know so far

Let's recall where we are.

We have seen some strange behaviors of infinite sets:

Example 15.1.1. Let $X = \mathbb{N}$, and let $Y \subset X$ be the collection of only *positive* natural numbers.

Then there exists a bijection from X to Y. For example, the function $f : X \to Y$, $f(n) = n + 1$ is a bijection.

In other words, even though Y is a subset of X, there is a bijection from X to Y. (And hence, by last time, a bijection from Y to X as well.)

Remark 15.1.2. Remember that "existence of a bijection" is the precise way we can articulate when two sets "have the same size." So the above example, which we've seen before, shows that "being a subset" is not at all the same thing as "being smaller," at least in the setting of infinite sets.

So a natural next question is: Are there infinite sets of different sizes? For this, we saw:

Theorem 15.1.3 (Cantor's Theorem)**.** For any set X, there is no bijection from X to $\mathcal{P}(X)$.

Corollary 15.1.4. There exist two infinite sets X and Y for which there exist no bijections from X to Y.

A *corollary* is a fact that follows “immediately” or “obviously”from another statement. In this case, the corollary is an immediate consequence of Cantor’s Theorem. Let’s see why:

Proof. Let X be any infinite set. (For example, you could take $X = \mathbb{R}$ or $X = \mathbb{N}$.) Then let $Y = \mathcal{P}(X)$. By Cantor’s Theorem, there is no bijection from X to Y. □

Remark 15.1.5. The word “immediately” is very subjective. It’s okay if the corollary isn’t obvious. Math is filled with corollaries that aren’t obvious, and we sometimes have to work hard to understand why it’s so obvious to the author, but not to the reader.

We are embarking into a wonderful new world. For example, we now have the following examples of sets:

$$\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathbb{R}, \mathcal{P}(\mathbb{R}).$$

Do we know which ones are bigger than others? Not yet.

And, we also have some ways of making new sets. We have seen that we can take

1. The union of a collection of sets, and

2. The intersection of a collection of sets.

Let’s see some more famous sets, and then another way to make new sets.

15.2 Some more famous sets: $\mathbb{Q}$ and $\mathbb{Z}$

But let’s also give names to other famous and important sets.

Notation 15.2.1. We let $\mathbb{Q}$ denote the set of all rational numbers.

Remark 15.2.2. Based on homework, we know two equivalent ways to think about rational numbers. A real number is called a rational number if it can be written as a quotient of two integers. Equivalently, a real number is called a rational number if its decimal expansion eventually begins to repeat some finite string of digits.

Example 15.2.3. The following are all elements of $\mathbb{Q}$:

$$0, \qquad 1, \qquad \frac{1}{2}, \qquad , \frac{11}{27}, \qquad \frac{4}{8}, \qquad \frac{-3}{8}.$$

Note that $\frac{1}{2}$ and $\frac{4}{8}$ are *equal* elements of $\mathbb{Q}$.

Notation 15.2.4. We let $\mathbb{Z}$ denote the set of all integers.

Example 15.2.5. The following are all elements of $\mathbb{Z}$:

$$\ldots, -100234, \ldots -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots, 1789, \ldots.$$

Example 15.2.6. We may write the following chain of subsets:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Remark 15.2.7. $\mathbb{Q}$ is used because Q stands for "quotient."

The letter $\mathbb{Z}$ is used because the German word for number is "zahl" and the word for numbers is "zahlen."

15.3 Some spoilers

So it's now natural to ask whether these sets admit bijections between each other. For example, is there a bijection from $\mathbb{N}$ to $\mathbb{Z}$? From $\mathbb{Z}$ to $\mathbb{Q}$? From $\mathbb{Q}$ to $\mathbb{R}$?

Let me give some stuff away. We'll try to prove this facts in the coming classes.

Theorem 15.3.1. There exist bijections from $\mathbb{N}$ to $\mathbb{Z}$, and from $\mathbb{N}$ to $\mathbb{Q}$.[1]

There also exists a bijection from $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$.[2]

It's quite an interesting theorem. For most of us, intuition would say that $\mathbb{Q}$ and $\mathbb{N}$ couldn't possibly admit a bijection between them. But we'll see a bijection (or the existence of a bijection) soon.

[1] In particular, it turns out we can think of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ as all having the "same size!"

[2] In particular, by Cantor's theorem, there is no bijection between $\mathbb{N}$ and $\mathbb{R}$.

15.4 The continuum hypothesis

Here's another natural question. If we know that $\mathbb{N}$ and $\mathcal{P}(N)$ are not in bijection, and so that (informally speaking) $\mathcal{P}(N)$ is "bigger" than $\mathbb{N}$, could there be a set of intermediate size? In other words, is there some set W so that W does not admit a bijection to $\mathbb{N}$ nor to $\mathcal{P}(\mathbb{N})$, but for which there exist bijections $\mathbb{N} \to W \to \mathcal{P}(\mathbb{N})$?

The *continuum hypothesis* is the assumption that there is no such W.

It turns out that we can *prove* that the existence of such a W is not provable from the usual axioms of set theory, nor disprovable. In fact, whether W exists is completely independent of the usual set-up of mathematics.[3] For many years, mathematicians were not sure whether they could prove or disprove the continuum hypothesis; now it is provable that those mathematicians would never have succeeded in proving or disproving it.

It's an amazing fact. It's like a "Truman show" moment—we are on a boat on the water, and suddenly we realize that what we thought was the horizon is just a painting on a wall. Our theory allows us to ask questions that the theory simply cannot answer.

So, at this point, it is a matter of religion or of philosophy to decide whether we take the continuum hypothesis as another axiom, whether we choose to not make it a part of our logical system at all, or whether to proceed with a logical system for which the falseness of the continuum hypothesis is taken as an axiom.

The world is your oyster. (Though, I am told that logicians and mathematicians often debate whether the continuum hypothesis should be considered true or not.)

Finally, you can ask a more general question. If X is an infinite set, is there always some "intermediary" set W so that there is an injection from X to W, and from W to $\mathcal{P}(X)$, but for which W is not in bijection with X nor with $\mathcal{P}(X)$? It turns out that this question is just as unanswerable for arbitrary infinite sets X.

The "generalized continuum hypothesis" is the assumption that no such W exists. It is again a matter of religion, philosophy, or preference whether to create a system of math in which this hypothesis is true, or is false.

One amazing fact, however, is that the generalized continuum hypothesis

[3] For those of you who are curious, Google "continuum hypothesis is independent of ZFC."

(together with the usual axioms of set theory) imply that the axiom of choice must be true. We haven't learned about the axiom of choice yet, but it is such an amazing fact that I couldn't resist telling it to you.