Lecture 16

Complements, products, sets of functions

Here are two new ways of producing new sets out of old.

16.1 Complements

Let A be a set, and choose a subset $B \subset A$.

Definition 16.1.1. The *complement of* B *in* A is the set of all elements in A that are not in B.

Often, when A is implicit or understood, we will often just say "the complement of B."

Exercise 16.1.2. Let A be a set. If B is the empty set, what is the complement of B in A?

Exercise 16.1.3. What is the complement of A in A?

Exercise 16.1.4. Let $A = \mathbb{Z}$ and let B be the set of all odd integers. What is the complement of B in A?

Definition 16.1.5. Any number in the complement of $\mathbb{Q}$ in $\mathbb{R}$ is called an irrational number.

The complement of $\mathbb{Q}$ in $\mathbb{R}$ is called the set of all irrational numbers.

Remark 16.1.6. As usual, I've introduced concepts before using notation. One way to write the complement of B in A is as follows:

$$\{x \in A \,|\, x \notin B\}.$$

We will use complements often, so we have notation for complements:

Notation 16.1.7. The complement of B in A is commonly denoted by any of the following notation:

- $A - B$. (This is Hiro's least favorite, as it is a bit too suggestive of subtraction.)
- $A \setminus B$.

When A is implicit, or understood, we often write

$$B^C$$

for the complement of B.

Warning 16.1.8. In some textbooks, B is not required to be a subset of A. You can still define the complement of B in A in this situation, again as the set of all elements of A not contained in B. So you might often see something like

$$A \setminus B = \{a \in A \,|\, a \notin B\}$$

even when B is not a subset of A.

Regardless of what textbook you are using, the complement of B in A is always equal to the following set:

$$A \setminus (B \cap A).$$

16.2 Products

Definition 16.2.1. Let X and Y be sets. Then the *Cartesian product*, or the *direct product*, or the *product* of X and Y is the set of all "ordered pairs"

$$(x, y)$$

for which $x \in X$ and $y \in Y$.

We will give a better meaning to ordered pair in a moment. Let's see some examlpes.

Example 16.2.2. Let $X = \{1, 2\}$ and $Y = \{a, b, c\}$. Then the product of X and Y consists of the following elements:

$$(1, a), \quad (1, b), \quad (1, c), \quad (2, a), \quad (2, b), \quad (2, c).$$

Example 16.2.3. The product of Y and X (note the switch in order of Y and X from the previous example) consists of the following elements:

$$(a, 1), \quad (b, 1), \quad (c, 1), \quad (a, 2), \quad (b, 2), \quad (c, 2).$$

Warning 16.2.4. Note that the elements $(a, 1)$ and $(1, a)$ are different! (The former is in the product of Y and X, while the latter is not.) So, the order matters. This is the sense in which "ordered pair" is used in Definition 16.2.1.

Remark 16.2.5. Let X be a finite set with n elements, and let Y be a finite set with m elements. Then the product of X and Y is another finite set, and it has $n \times m$ elements.

This motivates why we call this the "product" of X and Y.

Warning 16.2.6. A set contains far more information that just the number of elements it has—so, as usual, do not just think of "size" as the only important thing about a set!

That would be as though you consider two people the same if they have the same number of atoms in their body. There's a lot more to people than that!

In many important examples, X and Y are the same set:

Example 16.2.7. Let $X = Y = \mathbb{R}$. Then the product of X and Y contains elements that look like

$$(0, 0), \quad (1, 4), \quad (1, 1), \quad (\pi, e), \quad (-\sqrt{2}, 2), \quad (-\sqrt{2}, \pi^2),$$

et cetera. In fact, you are very used to the product of $\mathbb{R}$ and $\mathbb{R}$—the elements above also express points on the x-y plane.

So, the x-y plane that you are used to drawing from precalculus and calculus is a drawing of the product of $\mathbb{R}$ with itself.

As usual, I have introduced the concept before introducing the notation for the concept. Here is the notation:

Notation 16.2.8. Let X and Y be sets. We denote the product of X and Y by the notation
$$X \times Y.$$
When $X = Y$, we may also sometimes write
$$X^2$$
instead of $X \times X$.

Warning 16.2.9. However, it is common that we only write X^2 when $X = \mathbb{Z}$ or $X = \mathbb{R}$. (So we write $\mathbb{Z}^2$ or $\mathbb{R}^2$ instead of $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{R} \times \mathbb{R}$.) When X is an arbitrary set, we often do not write X^2, and explicitly write $X \times X$ for the product of X with itself.

Example 16.2.10. $\mathbb{R}^2$ is the set of all pairs (x, y) where x and y are real numbers. In other words, $\mathbb{R}^2$ is a set we often draw as the x-y plane.

We can take products of more and more sets.

Example 16.2.11. Let W, X, Y be sets. Then $W \times X \times Y$ is the set of all ordered triplets
$$\{(w, x, y) \,|\, w \in W, x \in X, y \in Y\}.$$
And, in general, for any $n \geq 1$, X^n is the set of all ordered n-tuples of elements of X.

16.3 Sets of functions

Notation 16.3.1. Let X and Y be sets. We let Y^X denote the set of all functions from X to Y.

Example 16.3.2. Let $X = \{1, 2\}$ and let $Y = \{\text{alice}, \text{bob}\}$. Here are all the possible functions f from X to Y:

- $f(1) = \text{alice}$ and $f(2) = \text{alice}$
- $f(1) = \text{alice}$ and $f(2) = \text{bob}$

- $f(1) = \text{bob}$ and $f(2) = \text{alice}$
- $f(1) = \text{bob}$ and $f(2) = \text{bob}$

Example 16.3.3. Let $X = \{1, 2\}$ and let $Y = \{\text{alice}, \text{bob}, \text{coco}\}$. Here are all the possible functions f from X to Y:

- $f(1) = \text{alice}$ and $f(2) = \text{alice}$
- $f(1) = \text{alice}$ and $f(2) = \text{bob}$
- $f(1) = \text{bob}$ and $f(2) = \text{alice}$
- $f(1) = \text{bob}$ and $f(2) = \text{bob}$
- $f(1) = \text{coco}$ and $f(2) = \text{alice}$
- $f(1) = \text{coco}$ and $f(2) = \text{bob}$
- $f(1) = \text{alice}$ and $f(2) = \text{coco}$
- $f(1) = \text{bob}$ and $f(2) = \text{coco}$
- $f(1) = \text{coco}$ and $f(2) = \text{coco}$

Remark 16.3.4. Suppose that X and Y are finite sets, say with a elements in X, and b elements in Y. Then to define a function, for every element of X, we must choose one of b elements in Y to assign. So the total number of possible assignments is

$$b \times b \times \ldots \times b$$

(where there are a copies of b in the product).

In other words, the size of the set of all functions from X to Y is given by

$$b^a.$$

You can see why some people chose to use the notation Y^X for the set of all functions from X to Y.

Warning 16.3.5. As usual, the expression "b^a" (for the size of Y^X) must not be taken too literally when X or Y are sets of infinite size.

Example 16.3.6. If X is the empty set, Y^X consists of exactly on element.

This can be rather confusing, so let's walk through it. We've done this before, but it doesn't hurt to do it again.

A function f from X to Y is an assignment—for every element $x \in X$, we must assign an element of Y called $f(x)$.

Imagine X is the set of students in a gym, and Y is some set of colors. You are asked by the gym teacher to go into the gym, and assign each student a color (so they can divide up into teams and play dodge ball or some crazy thing). You enter the gym. If you see no students there (so that X is the empty set), you can proudly say you are finish. You have assigned every student in the gym to a color.

So, there is a one, and exactly one, function from the empty set to any set.

This lines up with the formula b^a for the size of Y^X, by the way. If $X = \emptyset$, then $a = 0$, so $b^a = b^0 = 1$.

Example 16.3.7. When $Y = \emptyset$, and if X is non-empty, then Y^X is the empty set. That is because there are no functions from a non-empty set to an empty set. (There are no labels to assign!)

16.4 Exercises

Exercise 16.4.1. For the following examples of A and B, write out $A \setminus B$.

(a) $A = \{0, 1, 2, 3\}$ and $B = \{1, 3\}$.

(b) $A = \{0, 1, 2, 3\}$ and $B = \{0, 3\}$.

(c) $A = \{0, 1, 2, 3\}$ and $B = \{0, 1, 2, 3\}$.

(d) $A = \{0, 1, 2, 3\}$ and $B = \emptyset$.

How many elements are in A, in B, and in $A \setminus B$ in the above examples?

(You can see why some people use the notation $A - B$. However, be careful, as usual: What if A and B are both infinitely large?)

Exercise 16.4.2. For the following examples of A and B, write out $A \setminus B$.

(a) $A = \mathbb{N}, B = \{x \in \mathbb{N} \,|\, x > 0\}$.

(b) $A = \mathbb{Z}, B = \{x \in \mathbb{Z} \,|\, x > 0\}$. (You can give your answer in written words, or using set notation.)

(c) $A = \mathbb{R}, B = \{x \in \mathbb{R} \,|\, x > 0\}$.(You can give your answer in written words, or using set notation.)

(d) $A = \mathbb{R}, B = \mathbb{Q}$.

(e) $A = \mathbb{Z}, B = \mathbb{N}$.

Exercise 16.4.3. Let $W = \{1, 2, 3\}$, $X = \{1, 2\}$, $A = \{a, b, c\}$.

Write out the following sets:

(a) $W \times X$

(b) $X \times W$

(c) $W \times A$

(d) $X \times A$

(e) $(W \setminus X) \times A$

(f) $(W \times A) \setminus (X \times A)$

Exercise 16.4.4. Let X and Y be sets, and let $\underline{2} = \{1, 2\}$.

Let $A \subset \underline{2}^{X \cup Y}$ be the subset of those functions for which $f(1) \in X$ and $f(2) \in Y$.

Exhibit a bijection between A and $X \times Y$.

Exercise 16.4.5. Using the same notation as the previous problem, exhibit a bijection between $X^{\underline{2}}$ and X^2. (This is not a typo—make sure you know what each notation means!)

Exercise 16.4.6. Let X, Y, Z be sets.

(a) Exhibit a bijection between $X \times Y$ and $Y \times X$.

(b) Exhibit a bijection bewteen $(X \times Y)^Z$ and $X^Z \times Y^Z$.

16.5 True or False

Exercise 16.5.1. Which of the following is a true statement?

(a) Cantor's Theorem shows that there is no such thing as a largest set.

(b) Cantor's Theorem shows that the power set is the largest set.

(c) There exist two sets that (i) do not admit a bijection between them, and (ii) contain infinitely elements.

(d) There exists a bijection from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$.

(e) There exists a bijection from $\emptyset$ to $\mathcal{P}(\emptyset)$.

Exercise 16.5.2. Which of the following is a true statement?

(a) For any set X, there always exists an injection from X to $\mathcal{P}(X)$.

(b) For any set X, there always exists a bijection from X to $\mathcal{P}(X)$.

(c) For any set X, there always exists a surjection from X to $\mathcal{P}(X)$.

(d) For any set X, there always exists at least one function from X to $\mathcal{P}(X)$.