Lecture 17

Some facts about bijections; ifs

We're going to build toward the following theorem:

Theorem 17.0.1. There exists a bijection from $\mathbb{N}$ to $\mathbb{Q}$.

Remember, I made the claim in previous classes that $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ all admit bijections between them, but there is no bijection from any of these three sets to $\mathbb{R}$.

There are many ways of proving the above theorem; I'm going to prove it *not* by making an explicit bijection, but by invoking some generally useful statements about injections, surjections, and bijections.

17.1 Bijections as something transitive

Before we get to proving the theorem, I'd like to set some context. Consider the following statement:

"if A has the same size as B, and if B has the same size as C, then A has the same size as C."

This is probably intuitively true to you. But as of now the statement is informal, so let's turn it into the mathematics we know how to articulate:

Proposition 17.1.1. If there exists a bijection from A to B, and if there exists a bijection from B to C, then there exists a bijection from A to C.

This proposition confirms our intuitions about size, and that "admitting a bijection" at least seems consistent with the notion of "being equal in size."

Proof. Let $f : A \to B$ and $g : B \to C$ be bijections. I claim that $g \circ f$ is a bijection.

To show this, we must show that $g \circ f$ is both an injection and a surjection.

Well, f is an injection and g is an injection (because each is a bijection). And we proved in homework that the composition of two injections is an injection. So $g \circ f$ is an injection.

I also proved in a previous lecture notes that the composition of two surjections is a surjection. So $g \circ f$ is also a surjection.[1] This completes the proof. □

Remark 17.1.2 (Transitivity)**.** You might remember from a past mathematical life that inequalities satisfy a "transitive" property:

$$\text{If } x \leq y \text{ and } y \leq z, \text{ then } x \leq z.$$

Equality satisfis a transitive property too:

$$\text{If } x = y \text{ and } y = z, \text{ then } x = z.$$

Many relations in life satisfy this transitive property. We've just added to the list: We've seen that the notion of "admitting a bijection" is also transitive.

Example 17.1.3. Suppose X admits a bijection to Y, but that X admits no bijection to Z. Then Y admits no bijection to Z. (You can prove this by a proof by contradiction, using the transitive property of admitting bijections.)

"Admit a bijection" is a clunky phrase. We'll now introduce the following terminology. The word "cardinality" has a more precise meaning we'll encounter later in our course.

Definition 17.1.4. Let X and Y be sets. We say that X "has the same cardinality as Y" if there exists a bijection from X to Y.

You can think of "cardinality" as a fancy word for size. We'll see what it actually means later in this course.

[1] Note f and g are each surjections, because the are each bijections.

17.2 An injection one way guarantees a surjection the other way

Intuitively, an injection from X to Y seems to tell us that Y is "at least as large as X." Likewise, a surjection from Y to X seems to tell us the same thing. The following formalizes this intuition:

Proposition 17.2.1. Suppose that X and Y are non-empty sets. Then there exists an injection from X to Y if and only if there exists a surjection from Y to X.

Remark 17.2.2. For some of us, this may be the first time we have seen the phrase "if and only if." It is often abbreviated by the letters "iff."

What does this mean? Let's think through it logically. "p is true iff q is true" means "p is true if q is true, and p is true *only if* q is true."

In other words, p is true exactly when q is also true, and vice versa.

Such p and q are called "equivalent" statements. This is because the only situations in which p is true are exactly the only situations in which q is true.

To prove that p and q are equivalent statements, it suffices to prove two things: That p implies q, and that q implies p.

In other words, it suffices to prove "if p then q" and "if q then p."

Remark 17.2.3 (Converse)**.** Given the logical statement "if p then q," the statement "if q then p" is called the *converse* of the original statement.

Thus, p and q are equivalent exactly when "p implies q" is true, and when the converse is true.

Proof of Proposition 17.2.1. Suppose that there exists an injection from X to Y. Call it f. Choose some element $x_0 \in X$ (which we can do because X is assumed non-empty) and define a function g from Y to X as follows:

$$g(y) = \begin{cases} \text{The unique } x \text{ for which } f(x) = y & \text{if } y \in \text{image}(y) \\ x_0 & \text{if } y \notin \text{image}(y). \end{cases}$$

It is straightforward to see that g is a surjection.[2] So we have shown that if there is an injection from X to Y, then there is a surjection from Y to X.

[2] Choose some $x \in X$; we must exhibit some $y \in Y$ for which $g(y) = x$. For this, just set $y = f(x)$; by the definition of g, we see that $g(y)$ indeed equals x.

Let us now show the converse—that if there exist a surjection from Y to X, then there exists an injection from X to Y. Let $g : Y \to X$ be any surjection. Then for every $x \in X$, there exists some y_x for which $g(y_x) = x$. For every $x \in X$, choose such a y_x, and let $f(x) = y_x$.

To see that f is an injection, suppose that $f(x) = f(x')$. Then $y_x = y_{x'}$, so $g(y_x) = g(y_{x'})$. However, by definition of y_{blah}, we see that $x = g(y_x)$ and $x' = g(y_{x'})$. Putting the equalities in this paragraph together, we see that $x = x'$.

This completes the proof. □

Remark 17.2.4. In proving the converse, we had to *choose* some y for every x. It isn't entirely obvious that we can do this—well, actually, this is a litmus test for what kind of mathematician you are. Even among the most sophisticated professionals, it's either "intuitively obvious" or "not at all obvious" that one can make choices like this.

As it turns out, even on top of the usual axioms of set theory, we actually need an additional axiom in mathematics to allow us to choose the way we needed to choose in the previous proof. The fact that we can—given an arbitrary collection of sets—always choose an element from each set and the collection while remembering which collection it came from, is called the *axiom of choice*.

Note that the previous proof didn't give us an explicit injection. We saw that we could "make some choices" to construct an injection. This is typical of proofs that employ the axiom of choice—we end up with things while having no grasp of how to actually construct the things.

Mathematicians (like me) who are okay with such proofs are called "non-constructivist" mathematicians. Some mathematicians are constructivist, which means they only utilize proofs which explicitly construct the things asserted to exist.

I would advocate that these two different schools of thought are just as philosophical as they are scientific.

17.3 When two statements are equivalent

We have encountered the language of "if and only if," so I'd like to explicate it.

One of our biggest homework assignments says that two definitions of rational numbers are "equivalent."

In the assignment, we study two properties one could ask of a real number:

(a) The number can be expressed as a fraction of two integers.

(b) The number's decimal expansion eventually repeats some finite string of digits.

What we saw is that "if a number satisfies (a), it satisfies (b)" *and* "if a number satisfies (b), then it satisfies (a)."

In other words, a number satisfies both (a) and (b), or neither.

In homework, we interpreted this to mean that the set of numbers satisfying (a) is the *same* as the set of numbers satisfying (b).

Today, I'd like to talk about the logical structure of such a situation. Indeed, instead of thinking about two sets as being equal, we can speak of two logical statements as being *equivalent*.

17.3.1 If

Math is all about *if* statements. For example "If a prime number is bigger than 3, then it is odd." So it'll be important for us to understand what it means to use the word "if."

The word "if" sets up the hypotheses for something to be valid. For example, the statement *all animals have wings* is false. But it is true under some conditions. *If an animal is a bird, then it has wings.* There may be many conditions that make it true: *If an animal is a bat, then it has wings.*

And let's be explicit about what "if ... then ..." statements actually tell us in math. A phrase of the form "if p then q" is a short way of saying "if p is true, then q must be true."

Not every if-then statement is true. For example, here is a mix of true and false examples:

1. If I am a human being, I can walk. (This is a false statement. There are plenty of human beings who cannot walk.)

2. If p is an integer, it must be a prime number. (This is a false statement. There are plenty of integers that are not prime numbres.)

3. If p is a prime number, it is an integer. (This is a true statement.)

4. If p is a prime number and if p is even, then p is 2. (This is a true statement.)

As you can see, sometimes we omit the "then" in if-then statements. That's okay.

Finally, in English, it is common to retain the logical meaning of an if statement even after changing the order of words. For example, the following mean the same thing:

1. The train will be late if it does not leave on time.
2. If it does not leave on time, the train will be late.
3. If the train does not leave on time, then the train will be late.
4. If the train does not leave on time, the train will be late.

More abstractly, the following mean the same thing:

1. If p then q.
2. q if p.
3. p implies q.

Notation 17.3.1. Given two statements p and q, we let

$$p \implies q$$

denote the statement "if p then q." Equivalently, $p \implies q$ means "p implies q."

Note that $p \implies q$ is just another statement. It may true and it may be false.

Example 17.3.2. "n is prime $\implies$ n is odd" is a false statement.
"n is prime and larger than 3 $\implies$ n is odd" is a true statement.

17.3.2 Only if

Now, in everyday life, we sometimes come across the phrase "only if." For example, "You can run for President only if you are at least 35 years old."

Let's be careful about what this means. "Only if" does *not* mean if.

Consider: "You can run for President if you are at least 35 years old." This is a false statement, because to run for President, you must also have been born in the United States. Indeed, “You are at least 35 $\implies$ You can run for President” is a false statement.

Consider: “You can run for President only if you are at least 35 years old.” This is a true statement.

Warning 17.3.3. Be careful: In everyday life, some people use “only if” to imply “if.” In the usual logical interpretation of the term “only if,” this is an incorrect use.

In fact, “p only if q” has the exact same truth value as the following statements:

(a) “p is only true in situations where q is also true.”

(b) “if p is true, it must be that q is true.” (Because p can only be true only when q is true.)

(c) “if p then q.” (By the above, if p is true, then it must be that q is true—because p can only be true when q is true.)

(d) “if q is not true, then p must be not true.”

17.3.3 If and only if

Putting it altogether, to say

“p if q” and “p only if q”

is to say $q \implies p$ and $p \implies q$. In other words, p and q can only ever be true at the same time. They are either both true, or both false. This is what we mean when we say that p and q are logically equivalent.

Definition 17.3.4. We say that two statements p and q are *equivalent* if $p \implies q$ and $q \implies p$. When p and q are equivalent, we write

$$p \iff q.$$

This symbol is read aloud:

p if and only if q.

Of course, "p $\iff q$" is just a new statement. We say that p is equivalent to q if "p $\iff$ q" is a true statement.

Example 17.3.5. The following are all true statement.

1. Let T be a triangle. Then T has two sides of equal length if and only if T has two angles of equal size.

2. Let N be a natural number. Then N ends in 0, 2, 4, 6, or 8 if and only if N is divisible by two.

3. Let x be a real number. Then $x \geq 0$ if and only if there exists a real number y for which $y^2 = x$.

In particular, the above illustrate examples of equivalent statements. ("T has two sides of equal lengths" if equivalent to "T has two angles of equal size.")

17.3.4 Converses

So, p is equivalent to q when $p \implies q$ AND $q \implies p$.

Definition 17.3.6. We say that $p \implies q$ is the *converse* to $q \implies p$.

It is often the case that an if-then statement is true, but that its converse is false. (This is what makes equivalent statements special!)

Example 17.3.7.

1. Let R be a four-sided shape. "If R is a square, then it has four right angles" is a true statement. But its converse: "If R has four right angles, then it is is a square" is a false statement.

2. "If I have three trillion dollars, then I have more money than the state of North Dakota" is a true statement. But its converse, "If I have more money than the state of North Dakota, then I have three trillion dollars" is a false statement.

Remark 17.3.8. It is a very common mistake in life that human beings take a true if-then statement, and assume its converse to be true.

17.3.5 Revisiting our main result

Let's just look at our main result from today again. We can succinctly state it as follows. Let X and Y be non-empty sets. Then

$$\text{There exists an injection } X \to Y \iff \text{There exists a surjection } Y \to X.$$

17.4 Exercises

Exercise 17.4.1. Write the converse to each of the following statements.

(a) If a banana is delectable, then the pineapple wins.

(b) If R is a triangle, then it has more than two sides.

(c) If the cyclops is awake, then Odysseus is in trouble.

Exercise 17.4.2. Let X, Y, Z be non-empty sets. Suppose X admits an injection to Y, and Z admits a surjection to Y. Prove that X admits an injection to Z.

Exercise 17.4.3. Suppose p, q, r are three statements. Suppose that $p \implies q$, $q \implies r$, and $r \implies p$. Are the three statements equivalent?

What if $p_1, p_2, \ldots, p_n$ is a collection of n statements, and you know that $p_i \implies p_{i+1}$ $i = 1, 2, \ldots, n-1$, and that $p_n \implies p_1$. Are the n statements equivalent?