Lecture 18

Cantor-Schröder-Bernstein Theorem

18.1 An injection both ways guarantees a bijection

Now, what if we have an injection $f : X \to Y$, and some injection $g : Y \to X$? Intuitively, this would suggest that "Y is at least as big as X," and that "X is at least as big as Y." This would further suggest that Y and X ought to have the same size—i.e., that there ought to exist a bijection from X to Y.

We will prove that this is indeed the case:

Theorem 18.1.1 (Cantor-Schröder-Bernstein)**.** If there exists an injection from X to Y, and an injection from Y to X, then there exists a bijection from X to Y.

Proof. We learned this proof from Eugene Curtin.[1]

Think of this as a sketch of a proof, and not the actual proof. It will give you an idea of why the theorem is true.

The proof will be best visualized if you think of the notion of a "graph" in the sense of graph theory (not in the sense of calculus or algebra). To some mathematicians, a "graph" is just a collection of marshmallows (called

[1]However, there are some subtleties; you can treat this as a sketch of a proof. For this course, I'll be more interested that you *know* the theorem and how to use it, rather than how to prove it.

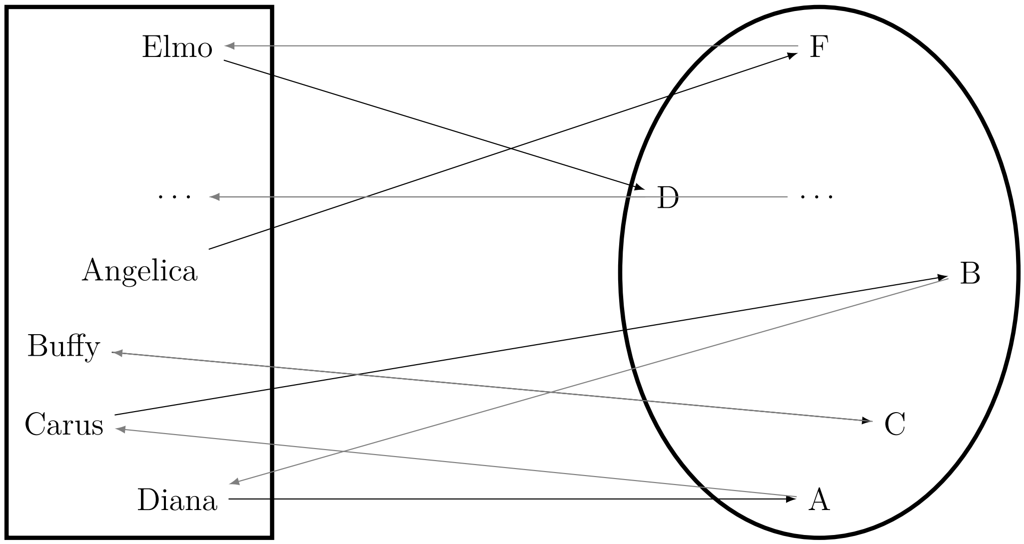

Figure 18.1:

vertices) and toothpicks (called edges), where the toothpicks have to have both ends shoved into the marshmallows.

So, suppose we have two sets X and Y, and an injection $f : X \to Y$ together with an injection $g : Y \to X$. We can make a graph out of this data as follows:

- There is a marshmallow/vertex for every element of Y, and for every element of X.
- For every $x \in X$, there is a toothpick with one end inserted into x, and the other end inserted into $f(x) \in Y$.
- For every $y \in Y$, there is a toothpick with one end inserted into y, and the other end inserted into $g(y) \in X$.

Figure 18.1 is a cartoon of the graph (marshmallow-toothpick configuration). The dark edges represent the toothpicks representing the relation $f(x) = y$, and the light gray edges represent the relation $g(y) = x$. (So for example, f sends Diana to A, while g sends A to Carus. Well, if you untangle the graph, you get three kinds of subgraphs. We will define a function $h : X \to Y$ using these subgraphs.

1. One subgraph is a "cycle," meaning a sequences of toothpicks that begins at some marshmallow, and (after some finite number of edges) returns to that same marshmallow. Notice that if you begin at a marshmallow labeled by an element $x \in X$, you necessarily need to use an even number of toothpicks to get back to x.[2] We will discard "every other" toothpick—meaning choose one to keep, discard the next toothpick, keep the third, discard the fourth, keep the fifth, and so forth. The cycle has been broken up into a bunch of disjoint edges now.

2. Another kind of subgraph is an "infinite chain open at both ends." As it turns out, there are as many edges as there are integers; so we can again choose to discard every other toothpick. For every toothpick you don't discard, one end of the toothpick is an element $x \in X$; the other end is an element $y \in Y$.

3. The third and final kind of subgraph is an "infinite chain open at one end, but beginning somewhere." If it begins at some $x \in X$, keep the toothpick starting at x, throw away the next (second) toothpick, keep the third toothpick, discard the fourth, and so forth. If the chain begins at some $y \in Y$, do the same thing. (Keep the toothpick emanating from y, discard the next, keep the third, discard the fourth, et cetera.)

You can tell this proof is rather informal. Regardless, we've managed to take our big, complicated marshmallow-toothpick graph, and discard "half" the toothpicks. The result is the same collection of marshmallows, but where each X-flavored marshmallow has exactly one toothpick going to exactly one Y-flavored marshmallow, and where every Y-flavored marshmallow is at the end of one of these toothpicks. For every X-flavored marshmallow x, define $h(x)$ to be the Y-flavored marshmallow on the other end of the toothpick coming out of x. Then h is a bijection. □

Remark 18.1.2. We have again used the axiom of choice. In particular, you're not supposed to have a great "feel" for what the bijection looks like. The important thing is the abstract result: If there are injections from X to Y, and from Y to X, then there is some bijection from X to Y.

[2] Likewise for elements of Y.