From the textbook "Introduction to Proof"

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Published by the Journal of Inquiry-Based Learning in Mathematics

September 2007

Chapter 1

Symbolic Logic – The thing that really separates us from the apes.

There was things which he stretched, but mainly he told the truth.
–Mark Twain, *Huckleberry Finn*

Somebody who thinks logically is a nice contrast to the real world.
–The Law of Thumb

How do humans communicate with each other? This question has many different possible answers. We are all certainly aware of the power of a facial expression or a hand gesture. However, each of these types of communication is often open to interpretation. One person may give the same hand gesture to two different people and those people may interpret it differently. This difference of interpretation may depend on many factors, too many in fact to go into here, but suffice it to say that mathematical proofs are not often constructed using facial expressions or hand gestures. Instead, we rely on written language. How then do we ensure that our language does not leave room for interpretation? This is sometimes a difficult chore, but in mathematical language we seldom run into this problem. The primary way we get around this is through the use of symbolic logic. According to Hale, *"Logic comprises the rules by which mathematicians operate, the "grammar" of the language."* [9] That is, logic is the tool we will use to create proofs. A proof is built up of statements, each of which follows from the preceding statement or statements.

1.1 Statements

Definition 1.1. *A* **statement** *is a declarative sentence which is either true or false, but not both.*

Note. Here we mean that the truth value of the statement does not depend on the person who is reading it. That is, it is possible for everyone to agree on the same truth value without regard to personal opinion.

Example 1.2. The following are examples of statements:

1. The Canadian national anthem is called *O Canada*.

2. The moon is made of green cheese.
3. Today is Wednesday.
4. James K. Polk was the 11th President of the United States.

Example 1.3. The following are not examples of statements.

1. Are you from Montana?
2. He is six feet tall.
3. Chocolate is better than vanilla.
4. Go west young man.

Question 1.4. Why are the items in Example 1.3 not examples of statements?

Exercise 1.5. Determine which of the following are statements. For the ones that are not statements, explain why not. If so, determine the truth value of the statement.

1. Calvin Coolidge was the greatest American President.
2. The square root of a rational number is always a rational number.
3. Mixing yellow and red paint will give you orange paint.
4. Life is like a box of chocolates.
5. When will the Red Sox win the World Series?
6. This sentence is false.
7. A group of owls is called a parliament.
8. Every former President of the United States is buried in the United States.

1.2 Compound statements and logical connectives

Now that we have an idea of what a statement is, we need to see how to put them together to form more complex statements and proofs. Then we will be in the position to discuss the rules of logic as they apply to compound statements, which are statements formed by simpler statements using logical connectives or implications. This gives us the following axiom about the construction of compound statements.

Axiom 1.6. *If A and B are statements then so are:*

1. Not A.	$(\sim A)$	(negation)
2. A and B	$(A \wedge B)$	(conjunction)
3. A or B	$(A \vee B)$	(disjunction)
4. If A, then B.	$(A \Rightarrow B)$	(implication)

Given that we can combine statements to form new statements, we need to figure out how to determine the truth or falsity of a compound statement. In mathematics there is seldom much to argue about with regard to truth, but how do we figure it out? We'll start with the simplest way to form a new statement, the negation. As the name suggests, the negation of a statement has the opposite truth value. Here is the *truth table* for a negation.

A	$\sim A$
T	F
F	T

Since we're not focused on symbolic logic as a core of this course, we will restate this as an axiom.

Axiom 1.7. *If A is a statement with a given truth value, then $\sim A$ is a statement with the opposite truth value.*

Exercise 1.8. Write the negation of the following phrases.

1. Pi is a positive real number.
2. Georgia is the eleventh largest state.
3. Flatland State University has no major in paleontology.

It will be worth your while to start thinking about how to form the negation of a statement. In later sections we will see that the negation of a statement can sometimes be a helpful thing when trying to prove the statement. Once we have more machinery, we will be able to create more interesting examples of negation that may come in handy. Stay tuned.

Now we'll move on to the logical connectives **and** and **or**. We again use truth tables to define the truth of these compound statements from the truth of their constituent parts.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Notice the structure of the truth tables. The statement "A and B" is true only when both A and B are true. That is, an **and** statement is false if either part is false. Meanwhile, the statement "A or B" is true if either part of the statement is true. In spoken language, it is often the case that an **or** statement will only be true when one part or the other is true, but not both. For example in a restaurant there may be a menu option for `Soup or Salad`. This would imply that you could have one or the other, but probably not both without paying extra. This kind of connective is called an *exclusive or* and is commonly used in computer science. We will not use this connective in mathematical proofs.

For the sake of completeness and consistency, we include axioms that correspond to the truth tables for the logical connectives **and** and **or**, just as we included the truth table for negation.

Axiom 1.9. *Let A and B be statements. Then the statement $A \wedge B$ is true if and only if both A and B are true.*

Axiom 1.10. *Let A and B be statements. Then the statement $A \vee B$ is true unless both A and B are false.*

Exercise 1.11. Determine what information would be necessary to determine whether each of the following statements is true or false. Then, determine the truth values.

1. The capital of Spain is Barcelona and today is Wednesday.
2. The capital of Spain is Barcelona or today is Wednesday.
3. Yao Ming is short and Verne Troyer is tall.
4. The ratio of the circumference of a circle to its diameter is 3 or the capital of Indiana is Indianapolis.
5. Water boils at 100° Celsius and freezes at 0° Fahrenheit.
6. A boar is a kind of a pig and an uninteresting person.

Now we're ready to attempt to make our first proof. The following is a pair of statements that are known as DeMorgan's Laws. They display the relationship between negation and the logical connectives *and* and *or*. You may want to think of them as symbolic logic analogues of the distributive property that you are familiar with, though this metaphor is not quite precise. You will see other analogues of the distributive property very soon. In the exercises that follow, the instructions are to show that two statements are *logically equivalent*. By logically equivalent, we mean that two compound statements have the same truth values for each choice of truth value for the statements that make up the compound statement. In order to show this, you will need to construct a truth table. Once we have these new tools in place, we will be in a position to give the constituent statements a mathematical context and begin proving mathematical statements.

Problem 1.12 (DeMorgan's Laws). Let A and B be statements and establish the following:

1. $\sim (A \wedge B)$ is logically equivalent to $(\sim A) \vee (\sim B)$
2. $\sim (A \vee B)$ is logically equivalent to $(\sim A) \wedge (\sim B)$

Before we move on to other logical connectives we'll collect a few "algebraic" properties of the connectives we already have.

Problem 1.13 (The Commutative Property). Prove each of the following:

1. $A \wedge B$ is logically equivalent to $B \wedge A$
2. $A \vee B$ is logically equivalent to $B \vee A$

Problem 1.14 (The Associative Property). Prove each of the following:

1. $(A \wedge B) \wedge C$ is logically equivalent to $A \wedge (B \wedge C)$
2. $(A \vee B) \vee C$ is logically equivalent to $A \vee (B \vee C)$

Problem 1.15 (The Distributive Property). Prove each of the following:

1. $A \wedge (B \vee C)$ is logically equivalent to $(A \wedge B) \vee (A \wedge C)$
2. $A \vee (B \wedge C)$ is logically equivalent to $(A \vee B) \wedge (A \vee C)$

1.3 Implications

Now we'll examine compound statements of the form "if A, then B." In this statement the "if A" part is called the **antecedent** or **hypothesis** and A is called the **sufficient condition** for the implication. (Why?) The "then B" part is called the **consequent** or **conclusion** and B is called the **necessary condition** for the implication. (Why?) In mathematical proofs we will encounter this statement form often. The truth table for an implication is as follows:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

It seems to be counterintuitive that an implication could be true if the antecedent is false. Here's an example that may help you see why we might want to construct our truth table this way:

> Suppose that your uncle Ted tells you that if you get an A in this class, then he will pay for your next semester's tuition. If you get an A and he pays, then he told you the truth. On the other hand, if you get an A and he does not pay, then you can conclude that he lied to you. But if you do not get an A, then whether he pays or not, in neither case can you conclude that he lied to you. Thus, if we choose a truth table in which the antecedent of an implication being false goes with the statement form being true, then we are consistent with this idea.

As before, we now include an axiom that we can use when we are proving a statement in the form of an implication.

Axiom 1.16. *Let A and B be statements. Then the statement "If A, then B" is true unless A is true and B is false.*

Exercise 1.17. Determine the truth or falsity of the following statements.

1. If x is an integer, then $x^3 > 0$.
2. If y is a positive integer, then y can be written as a sum of powers of two.
3. If π is a rational number, then the area of a circle is $E = mc^2$.
4. If Homer Simpson is the President of the United States, then Marge Simpson is the Queen of England.
5. If Jupiter has more than three moons, then we live in the 20th century.