Problem 1.15 (The Distributive Property). Prove each of the following:

1. $A \wedge (B \vee C)$ is logically equivalent to $(A \wedge B) \vee (A \wedge C)$
2. $A \vee (B \wedge C)$ is logically equivalent to $(A \vee B) \wedge (A \vee C)$

1.3 Implications

Now we'll examine compound statements of the form "if A, then B." In this statement the "if A" part is called the **antecedent** or **hypothesis** and A is called the **sufficient condition** for the implication. (Why?) The "then B" part is called the **consequent** or **conclusion** and B is called the **necessary condition** for the implication. (Why?) In mathematical proofs we will encounter this statement form often. The truth table for an implication is as follows:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

It seems to be counterintuitive that an implication could be true if the antecedent is false. Here's an example that may help you see why we might want to construct our truth table this way:

> Suppose that your uncle Ted tells you that if you get an A in this class, then he will pay for your next semester's tuition. If you get an A and he pays, then he told you the truth. On the other hand, if you get an A and he does not pay, then you can conclude that he lied to you. But if you do not get an A, then whether he pays or not, in neither case can you conclude that he lied to you. Thus, if we choose a truth table in which the antecedent of an implication being false goes with the statement form being true, then we are consistent with this idea.

As before, we now include an axiom that we can use when we are proving a statement in the form of an implication.

Axiom 1.16. *Let A and B be statements. Then the statement "If A, then B" is true unless A is true and B is false.*

Exercise 1.17. Determine the truth or falsity of the following statements.

1. If x is an integer, then $x^3 > 0$.
2. If y is a positive integer, then y can be written as a sum of powers of two.
3. If π is a rational number, then the area of a circle is $E = mc^2$.
4. If Homer Simpson is the President of the United States, then Marge Simpson is the Queen of England.
5. If Jupiter has more than three moons, then we live in the 20th century.

6. If the Earth is the center of the universe, then $3 = 5$.

As in non-mathematical English, there is often more than one way to say what you want to say. The following problem gives us an additional way to express the statement "*if A, then B*." We will see more about alternative ways to express a statement later. Stay tuned.

Problem 1.18. Prove that the following statements are equivalent:

1. If A, then B. $[A \Rightarrow B]$
2. Not A, or B. $[\sim A \vee B]$

Problem 1.19. Prove that the following statements are equivalent:

1. It is not the case that A implies B. $[\sim (A \Rightarrow B)]$
2. A and not B $[A \wedge \sim B]$

In the mathematical literature, and in logic texts, there are lots of other statements that carry the same meaning as the statement "*if A, then B*." These include:

A suffices for B	A only if B	B is necessary for A	B if A

You should think about why these statement forms are equivalent and try to become comfortable with using different, equivalent statement forms more or less interchangeably.

There are three natural statements that are closely associated with an implication. We can create these by reversing the direction of the implication and/or adding negations. The three statements are the **converse**, **inverse** and **contrapositive** of the implication and are given by the following.

Definition 1.20. *Suppose that $A \Rightarrow B$. The* **converse** *of $A \Rightarrow B$ is $B \Rightarrow A$. The* **inverse** *of $A \Rightarrow B$ is $(\sim A) \Rightarrow (\sim B)$. The* **contrapositive** *of $A \Rightarrow B$ is $(\sim B) \Rightarrow (\sim A)$.*

Example 1.21. Form the converse, inverse and contrapositive of the following statements:

1. If it rains, then the ground gets wet.
2. If a polygon has no diagonals, then it is a triangle.
3. If a function is differentiable, then it is continuous.
4. If fren is a glurb, then ramal will qapla.

Note: We will not often deal with the inverse of a statement. It is included here for the sake of completeness.

Given an implication, what do you think the relationship is among the truth tables for its converse, inverse and contrapositive? We can verify that an implication and its contrapositive are logically equivalent, while the inverse and converse may have different truth values than the original implication. The fact that a statement and its contrapositive are logically equivalent will often be useful to us when constructing proofs. More on this later.

Problem 1.22. Prove that an implication and its contrapositive are logically equivalent.

Problem 1.23. For a given implication, prove that its inverse and converse are logically equivalent.

Sometimes an implication and its converse are both true. In this case, we can make the statement "A if and only if B," or "A iff B," or $A \Longleftrightarrow B$. We call the two statements logically equivalent and this type of statement form is called a **biconditional equivalence**.

1.4 Quantifiers

It isn't very often that mathematicians make statements about isolated objects or cases. Often, if not usually, the statements concern every one of a certain kind of object or a statement asserts that it is possible to find an example of a certain thing. This gives rise to the need for **quantifiers**. There are two of these phrases: the universal quantifier **for all** ($\forall$), and the existential quantifier **there exists** ($\exists$). The use of the universal quantifier implies that a certain property is true for every object in a certain class and the existential quantifier states that an object possessing a certain property exists. Note that when we assert the existence of something in mathematics, we are not saying that only one of them exists. For example, if we say "there exists a flarn", we are making no claims about the uniqueness of the flarn, just that there is *at least one of them*. If we want to say something about uniqueness, we need to say it directly, such as "there exists a unique tweddle flarn."

Example 1.24. The following are examples of statements with quantifiers. Determine the truth value of each statement. Now try to form the negation of each statement.

1. All mammals have hair.
2. All rational numbers are natural numbers.
3. There exists a man from West Virginia.
4. There exists a real number whose square is not positive.

What happens when we negate a statement with a quantifier? That is, what is the effect of negation on quantifiers? If we negate a statement with a *for all* in it, then we're saying that it's not true that every object in a certain class possesses a certain property. In order for the negated statement to be true, we need only find one object for which the property is not true. On the other hand, when we negate a *there exists* statement, we're saying that no object possessing a certain property exists. If we denote the given property by $P(x)$, then the relationship between **for all** and **there exists** works as follows:

Example 1.25. The following pairs of statements are logically equivalent.

1. (a) "$P(x)$ is true for all x"
   (b) "There is no x for which $P(x)$ is not true."
2. (a) "There is some x for which $P(x)$ is true."
   (b) "It is not the case that $P(x)$ fails for all x."

Put another way, in a sort of extended logical system, a **for all** statement is like the conjunction of a very large number of simpler statements and a **there exists** statement is like the disjunction of a very large number of simpler statements.

Exercise 1.26. Rephrase the preceding comment in your own words.

Exercise 1.27. Write the negation of the following phrases.

1. All squares are rectangles.
2. Some squares are rectangles.
3. No squares are rectangles.