

Lecture 22

Negation

22.1 Negation

If we have a statement P , we want to understand how to write the statement “not P .” This is written $\neg P$. Some people also write this as $\sim P$.

Example 22.1.1. If P is the statement “the sky is blue,” then $\neg P$ is the statement “the sky is not blue.”

If P is the statement “every dog has its day,” then $\neg P$ is the statement “there is some dog that never has its day.”

The last example, I hope, is a bit of an indication that we want to be able to write down negations in multiple ways. Just writing “Not every dog has its day” doesn’t always help. Instead writing “There is some dog that never has its day” says that the truth of $\neg P$ can be determined by exhibiting one dog with some property—one dog who never has its day.

As usual, the utility of all this formal logic notation is to break down complicated statements into simple ones. So let me introduce you to negations of the most common building blocks: \forall statements and \exists statements.

22.2 Negation of for all

Example 22.2.1. Let P be the statement “Everybody hurts.”¹ Put another way, P is the statement “For every person x , x hurts.” Or “ $\forall x$, x hurts.”

If we go further and say $H(x)$ is the statement “ x hurts,” then we may write

$$\forall x H(x).$$

Now, how do we write the negation of “everybody hurts” ? Well, the negation would be the statement “not everybody hurts.” Put another way, “there is somebody who does not hurt.” This is the important logical step. We have replaced “not everybody” with “there is somebody,” at the expense of negating “hurts.” Symbolically,

$$\neg(\forall x H(x)) = \exists x \neg H(x).$$

To make this easier to read, you won’t be punished for writing “such that”:

$$\neg(\forall x H(x)) = \exists x \text{ s.t. } \neg H(x).$$

Take-away. The negation of $\forall x H(x)$ is $\exists x \neg H(x)$. In other words, “ $H(x)$ is true for all x ” has negation “There is some x for which $H(x)$ is not true.”

Put more algebraically, if you think of \neg like multiplication by -1 , it turns \forall into \exists and any H into $\neg H$.

22.3 Negation of there exists

Example 22.3.1. Let P be the statement “There is somebody out there for you.”

The (depressing) negation of P is that “there is nobody out there for you.” Now, there are many ways to express this idea, and it turns out the most testable/verifiable way to express this idea, for us, will be to say: “Everybody out there is not for you.”

Put more symbolically, “there is somebody out there for you” can be rewritten as “ $\exists x F(x)$ ” where $F(x)$ is the statement “ x is a person for you.”

¹From Wikipedia: “*Everybody Hurts*” is a song by American rock band R.E.M. from their eighth studio album, *Automatic for the People* (1992), and released as a single in April 1993.

The negation is as follows:

$$\neg(\exists x F(x)) = \forall x \neg F(x).$$

In other words, every person x is not for you.

22.4 Negation of or and and

What would it mean for “I am eating Thai food or I am sad” to be negated (that is, for it to be false)? It would mean

“I am not eating Thai food and I am not sad.”

More generally,

$$\neg(p \vee q) = \neg p \wedge \neg q.$$

And what would it mean for “I am eating bacon and I am happy” to be negated? It would mean one of the two clauses must be false. That is,

I am not eating bacon *or* I am not happy.

More generally,

$$\neg(p \wedge q) = \neg p \vee \neg q.$$

22.5 Negation of implication

What would it mean for “If I am eating bacon, I am happy” to be false? It must be that it is possible for you to be eating bacon, and yet not be happy. More generally,

$$\neg(p \implies q) = p \wedge \neg q.$$

22.6 Summary of basic negations

Statement	Negation
$\forall x H(x)$	$\exists x \neg H(x)$
$\exists x F(x)$	$\forall x \neg F(x)$
$p \implies q$	$p \wedge \neg q$
$p \vee q$	$\neg p \wedge \neg q$
$p \wedge q$	$\neg p \vee \neg q$

22.7 A worked-out example

Here is a condition you have seen in calculus, or that you will see in analysis:

For all $\epsilon > 0$, there exists $\delta > 0$ for which $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$.

When the numbers a and L , and the function f , satisfy the above statement, we say that L is the limit of f at a .

Here is the power of what we've done. *Even if you have no understanding of what the above statement means*, we can write out its negation. In other words, we can understand what it means for L to not be the limit of f at a .

For example, the above statement can be re-written as

$$\forall \epsilon > 0, A(\epsilon)$$

where $A(\epsilon)$ is the long statement “there exists $\delta > 0$ for which $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$.”

Then we know the negation must be

$$\neg(\forall \epsilon > 0, A(\epsilon)) = \exists \epsilon > 0, \neg A(\epsilon).$$

In other words, there exists some $\epsilon > 0$ for which $A(\epsilon)$ is not satisfied.

So now we investigate $A(\epsilon)$ to understand its negation. $A(\epsilon)$ is the statement

“there exists $\delta > 0$ for which $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$ ”

which we can re-write as

$$\exists \delta > 0 B(\delta, \epsilon)$$

where B is some statement depending on δ and ϵ .² So we know that the negation is given by

$$\neg(\exists \delta > 0 B(\delta, \epsilon)) = \forall \delta > 0 \neg B(\delta, \epsilon).$$

And, plugging this in for $\neg A(\epsilon)$, we see that the negation can be written as

$$\neg(\forall \epsilon > 0 A(\epsilon)) = \exists \epsilon > 0 \forall \delta > 0 \neg B(\delta, \epsilon). \quad (22.7.0.1)$$

In other words, “There exists $\epsilon > 0$ such that for all $\delta > 0$, $B(\delta, \epsilon)$ is false.”

²Specifically, B is the statement that $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$

So now we are left to understand the negation of $B(\delta, \epsilon)$. Well, $B(\delta, \epsilon)$ is a statement of the form

$$\forall(x)C \implies D$$

where C is the statement $0 < |x - a| < \delta$ and D is the statement $|f(x) - L| < \epsilon$. You saw in previous lectures that the negation of $C \implies D$ is “ C and $\neg D$.”

$$\neg(C \implies D) = C \wedge \neg D.$$

Now, because D is the statement $|f(x) - L| < \epsilon$, we *for the first time in this example* will use what we know about real numbers: The negation of $|f(x) - L| < \epsilon$ is the inequality $|f(x) - L| \geq \epsilon$. That is,

$$\neg(|f(x) - L| < \epsilon) = |f(x) - L| \geq \epsilon.$$

Thus, putting the last few lines together, we see

$$\begin{aligned} \neg B(\delta, \epsilon) &= \neg(\forall(x)C \implies D) \\ &= \exists x C \wedge \neg D \\ &= \exists x C \wedge (|f(x) - L| \geq \epsilon) \\ &= \exists x(0 < |x - a| < \delta) \wedge (|f(x) - L| \geq \epsilon). \end{aligned}$$

Plugging this back into (22.7.0.1), we find that the negation we have been looking for is:

$$\exists \epsilon > 0 \forall \delta > 0 \exists x(0 < |x - a| < \delta) \wedge (|f(x) - L| \geq \epsilon)$$

In other words, the statement “ L is not the limit of f at a ” is equivalent to the statement “there exists some $\epsilon > 0$ such that for all $\delta > 0$, there is some x for which $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$.”

Take-away. Even if a statement is super-complicated, you can break it down into simple pieces to work out the negation of the statement. Of course, the negation is often as complicated as the statement you began with; but you can be confident it is correct.