Lecture 24

Making new sets: Equivalence relations and quotient sets

Today we're going to learn about how to take a set X, and then "identify" or "collapse" different elements of X as though they were equal. This involves (i) Saying what we mean by a rule for declaring various elements of X equivalent, and (ii) Constructing a new set that results from equating those elements (making those elements equal, not just equivalent).

We saw this idea arise last lecture, when we realized there is a surjection

$$q : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}, \qquad (a, b) \mapsto \frac{a}{b}.$$

I want you to pay attention to this surjection. It is telling us something that perhaps we already knew: You can think of a rational number simply as an ordered pair of integers (a, b).[1] (This is reflected in the fact that q is a surjection.) Of course, the choice of (a, b) is not unique, as $(2, 1)$ and $(6, 3)$ represent the same rational number[2]. (This is reflected in the fact that q is not an injection.)

24.1 Spreadsheets

At the start of every semester, I take a survey. Sometimes, a students fills out the survey multiple times. And I might get a resulting spreadsheet that looks like this:

[1] The order tells us that a is on top, and b is the denominator.

[2] Because $2/1 = 6/3$

Erica Berica
John Doe
Ralph Ralphson
john doe
Alejandra
Oyendamola
John doe
Doe john
ralph ralphson
erica berica

Clearly, John and Erica and Ralph filled out the survey multiple times. And as a result, I have a list of names that is longer than the actual list of names in class. So let's say I want to create a new spreadsheet with the correct number of students.

One way that you would probably do this is by deleting some of the repeated rows. Here are two possible outcomes of doing this:

Erica Berica
john doe
Alejandra
Oyendamola
ralph ralphson

Ralph Ralphson
john doe
Alejandra
Oyendamola
erica berica

Great. But this is frustrating, because two different people might end up deleting different rows. So even if those people end up with the same number of rows (five) the spreadsheets they end up making could literally be different.[3]

Is there a way to do get rid of redundancies without deleting rows? Yes, there is! Here is how we can think of "getting rid of redundancies" after today's class:

{ Erica Berica, erica berica }
{ john doe, John Doe, John doe, Doe john }
{ ralph ralphson, Ralph Ralphson }
{ Alejandra }
{ Oyendamola }

[3]This different is not merely in the order of the lines in the spreadsheet; ralph ralphson is literally different from Ralph Ralphson, for example due to capitalization.

The three lists above have the same number of rows: five. But rows of this third list do not consist of a name; every row consists of a *set of names*. So for example, the last row is not a name called Oyendamola, it consists of a *set* with one element in it (and this element happens to be the name Oyendamola). Likewise, the first row is a set consisting of two elements; the two elements are Erica Berica, and erica berica. The way we produced this list of five sets is by creating sets that consist of equivalent names.

There are advantages and disadvantages to these two approaches. (One approach is to create a new list by deleting rows; the other is to create a new list of sets by creating sets out of equivalent names.) The great advantage of the second approach is that there is a natural *function* from the original list of names to the final list of sets of names: You send a name to the set that it's contained in. For example,

$$\text{John Doe} \mapsto \{ \text{ john doe, John Doe, John doe, Doe john } \},$$

and

$$\text{Doe john} \mapsto \{ \text{ john doe, John Doe, John doe, Doe john } \}.$$

More generally, given a set X (for example, the set of names entered in a survey) we can create a rule about when to consider two elements of X to be equivalent. This rule is called an *equivalence relation*, and we will give a rigorous definition shortly.

Remark 24.1.1. There is a big difference between the word "equal" and the word "equivalent." In a set, two things are equal if they are literally the same thing—that is, the "two things" were actually one thing. But two things that are not equal may be "equivalent" from some perspective.

Example 24.1.2 (Of when we might want to consider two non-equal things to be equivalent)**.** For example, suppose we have two congruent triangles in the plane. The two triangles may not literally be equal (for example, their vertices may be at different points of the plane) but, depending on our purpose at the moment, we may want to consider these two unequal triangles to be equivalent. As another example, two similar triangles (i.e., having equal angle measures) may not be equal, but we may want to consider them as equivalent.

As another example, you might consider two numbers to be equivalent if they are both even, or if they are both odd. Clearly 2 and 4 are *not* equal, but they may be considered equivalent for certain purposes.

Or, you might consider pairs of integers (a, b) and (a', b'), and declare them to be equivalent if $ab' = a'b$.[4]

One point of emphasis is that the notion of equivalence is *up to us*. We can decide when we want to consider two things to be equivalent based on what is convenient in the moment. "Equivalent" is not some canonical notion, but rather a notion we must specify in each context.

As a final example, two sets may not be equal, but they may have the same cardinality (i.e., there may be a bijection between them).

Once we are given this rule, we can create a new set, which we will call $X/\sim$. (This is read "X mod tilde," or "X mod twiddle," or sometimes, X mod sim). An element of $X/\sim$ will be a *set* of elements of X. Any element of $X/\sim$ will be a set containing *all* elements of X that are equivalent to each other. There will be a natural function from X to $X/\sim$, taking an element of X to the set of elements equivalent to it.

This $X/\sim$ will be called a *quotient* of X by the equivalence relation $\sim$, and the function $X \to X/\sim$ is called the *projection map*.

24.2 Equivalence relations

When we declare a rule for treating certain elements of X as equivalent, we will encode this information in a subset $R \subset X \times X$. We will think of two elements x, x' as equivalent if and only if $(x, x') \in R$. Immediately, we see some natural things we should demand of R:

(i) Every element of X should be equivalent to itself.

(ii) If x is equivalent to x', then certainly x' should be equivalent to x.

(iii) If x is equivalent to x' and x' is equivalent to x'', then x should be equivalent to x''.

Following our intuitions laid out above, mathematicians have come upon the following definition.

Definition 24.2.1 (Equivalence relation)**.** Let X be a set. A subset $R \subset X \times X$ is called an *equivalence relation* on X if R satisfies the following propreties:

[4] This is exactly what it means for the fractions a/b and a'/b' to be equal!

(i) For all $x \in X$, the element (x, x) is in R.

(ii) For all $x, x' \in X$, if $(x, x') \in R$, then $(x', x) \in R$.

(iii) For all $x, x', x'' \in X$, if (x, x') and (x', x'') are in R, then (x, x'') is in R.

Notation 24.2.2. Sometimes we are lazy. Instead of writing out the whole sequence of symbols $(x, x') \in R$, we will write $x \sim x'$. In this notation, the above conditions can be re-written as

(i) For all $x \in X$, $x \sim x$.

(ii) For all $x, x' \in X$, if $x \sim x'$, then $x' \sim x$

(iii) For all $x, x', x'' \in X$, if $x \sim x'$ and $x' \sim x''$, then $x \sim x''$.

Notation 24.2.3. Likewise, we will sometimes refer to $\sim$ as an equivalence relation (rather than R). When you read the sentence "Let $\sim$ be an equivalence relation on X," you should understand that the notation $\sim$ actually represents the data of R.

Remark 24.2.4. In general, a "relation on X" is any subset of $X \times X$. In further classes, you may see examples of relations like orders, partial orders, total orders, linear orders, et cetera.

Warning 24.2.5. If you're reading a book and they don't define an equivalence relation on X as a subset of $X \times X$, you should be very careful about what definition the book is actually giving. One philosophy of our course is that any rigorous definition should ultimately be given in terms of sets and functions.

24.3 Examples of equivalence relations

Example 24.3.1 (Diagonal relation)**.** Let X be a set. The *diagonal* relation, or diagonal equivalence relation, is the set $\Delta = \{(x, x)\}$. that is, Δ consists of all pairs (x, x') for which $x = x'$. This is an equivalence relation.

(i) By definition, for every $x \in X$, we see that $(x, x) \in \Delta$.

(ii) If $x \sim x'$, then $x = x'$, so $x' = x$; in particular, $x' \sim x$.

(iii) If $x \sim x'$ and $x' \sim x''$, then $x = x'$ and $x' = x''$, so $x = x''$, hence $x \sim x''$.

This is, actually, a silly equivalence relation. It declares that x is equivalent to x' is and only if x is *equal* to x'.

Example 24.3.2. Let $X = \mathbb{Z}$. Let $R \subset \mathbb{Z}\times\mathbb{Z}$ be the of all pairs $(a, b) \in \mathbb{Z}\times\mathbb{Z}$ satisfying the condition that $b-a$ is divisible by 2. I claim R is an equivalence relation:

(i) For all $a \in \mathbb{Z}$, $a - a = 0$, which is divisible by 2. So indeed, $(a, a) \in R$.

(ii) If $a - b$ is divisible by 2, then so is $b - a = -(a - b)$. So $(a, b) \in R \implies (b, a) \in R$.

(iii) Note that $(c - a) = (c - b) + (b - a)$. If we know that both $c - b$ and $b - a$ are divisible by 2, then so is their sum—hence $c - a$ is divisible by 2. This shows that if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Example 24.3.3. Let X and Y be sets, and fix a function $f : X \to Y$. Define a relation $R \subset X \times X$ by declaring that for points $x, x' \in X$, $(x, x') \in R \iff f(x) = f(x')$. Then R is an equivalence relation.

(i) For any $x \in X$, we of course have $f(x) = f(x)$, so $(x, x) \in R$.

(ii) If $f(x) = f(x')$, then $f(x') = f(x)$.

(iii) If $f(x) = f(x')$ and $f(x') = f(x'')$, then of course $f(x) = f(x'')$.

Example 24.3.4. Let X be the set of all humans. Given two humans x and x', we will say that $x \sim x'$ if and only if x is related to x' (i.e., if they are related genetically or as family).

(i) Any person is related to themselves.

(ii) If person x is related to person x', then person x' is related to person x.

(iii) If person x is related to person x' and if person x' is related to person x'', then person x is related to person x''.

(Warning: The notion of being "related to" is informally familiar to us, but I am not giving a rigorous definition of family or genetic relation.)

Example 24.3.5 (Trivial relation)**.** Let X be a set. The *trivial* relation (sometimes called the universal relation) is the relation given by $X \times X$. So for any $x, x' \in X$, we have that $x \sim x'$. (Everything is equivalent.) The three properties of being an equivalence relation are verified straightforwardly.

Example 24.3.6 (Toward rational numbers)**.** Let $X = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ be the set of ordered pairs (a, b) where a, b are integers and $b \neq 0$.

Let $R \subset X \times X$ be the relation consisting of elements

$$R = \{((a,b),(a',b')) \text{ s.t. } ab' = a'b\}.$$

You should verify that R is an equivalence relation.

24.4 Equivalence classes

So an equivalence relation allows us to create a mathematical framework for when to think of certain elements in a set X as equivalent. (We think of x and x' as equivalent if and only if $(x, x') \in R$.)

Now let's see how to construct a set where we have "identified" equivalent elements of X. Informally, we are about to construct a new set where if x and x' are equivalent in X, then "$x = x'$" in this new set. (This equality is in "quotes" because x and x' will not be elements of this new set.)

Definition 24.4.1. Let R be an equivalence relation on X, and choose a subset $E \subset X$. We say that E is an *equivalence class* (or an *R-equivalence class*) if the following holds:

1. E has at least one element. (So E is non-empty.)

2. For any two elements $x, x' \in E$, we have that $x \sim x'$.

3. If $x \in E$ and x' is an element in X for which $x \sim x'$, then $x' \in E$.

Remark 24.4.2. Here is an example of what an equivalence class is. Suppose that you think of two elements x and x' as "in the same family" if $x \sim x'$. Then one equivalence class can be thought of as one family—everybody in E is in the same family, and everybody in that family is in E. Importantly, E is not just part of one family, nor does it contain members from multiple families.

Notation 24.4.3 ($X/\sim$)**.** Let X be a set and R an equivalence relation on X. Then we use the notation

$$X/\sim$$

to denote the set of all R-equivalences classes.

Remark 24.4.4. So $X/\sim$ is another "bag of bags" i.e., a set of sets. In fact, $X/\sim$ is a subset of the power set $\mathcal{P}(X)$.

In terms of the previous remark, you can think of $X/\sim$ as the collection of all families. So an element of $X/\sim$ is a family. Note that a family is a set–it contains members.

Notation 24.4.5 (Square brackets)**.** Let X be a set and R an equivalence relation. Given an element $x \in X$, we let $[x]$ denote the equivalence class to which x belongs. In other words,

$$[x] = \{x' \in X \mid x' \sim x\}.$$

Note that $[x] \in X/\sim$.

Warning 24.4.6. Note that even if $x \neq x'$, it may be that $[x] = [x']$. So be careful about the notation $[x]$; it is very convenient, but it can be difficult to remember which elements are contained in $[x]$. Remember that $[x]$ is a set—in fact, $[x] \subset X$.

On the other hand, the fact that $x \neq x'$ but we can have $[x] = [x']$ is exactly the manifestation of what we wanted: Two elements may not be the same but equivalent; in $X/\sim$, they become *equal* according to the rules we set out for equivalence.

Example 24.4.7. If $X = \mathbb{Z}$ and an equivalence relation is defined by $a \sim b \iff a - b$ is divisible by 2, then $X/\sim$ has exactly two elements. The two elements can be written as

$$\{\ldots, -4, -2, 0, 2, 4, \ldots\}, \qquad \{\ldots, -3, -1, 1, 3, \ldots\}.$$

In other words, the two sets are the set of all even numbers, and the set of all odd numbers.

Example 24.4.8. If X is a set and $R = X \times X$ is the trivial relation, then $X/\sim$ has exactly one element. (Informally, this is because the trivial relation declares every element to be equivalent, so once you "collapse" all of them, or identify every element, you are left with one thing. Or, there is one family, because everybody is related.)

24.5 The quotient map

Definition 24.5.1. Let X be a set and $\sim$ an equivalence relation. There is a function

$$q : X \to X/\sim, \qquad x \mapsto [x].$$

We call this the *quotient map*.

Example 24.5.2. If X is a set of people and we declare $x \sim x'$ if x and x' are related, let's say that x and x' are in the same *family* if they are related. Then $X/\sim$ is the set of families, and the quotient function $q : X \to X/\sim$ sends a person to the family they are a member of.

24.6 Exercises

Exercise 24.6.1. Let $X = \{a, b\}$. (This is a two-element set.) Write down all equivalence relations that this set admits, and write down all equivalences classes for each equivalence relation.

(You should be able to write two equivalence relations. One of them will give rise to one equivalence class, while the other will give rise to two equivalences classes.)

Let $X = \{a, b, c\}$. (This is a three-element set.) Repeat the above.

(You should be able to write five equivalence relations.)

Exercise 24.6.2. Let X be a set and $\sim$ an equivalence relation.

(i) Show that the quotient map $X \to X/\sim$ is a surjection.

(ii) Show that if E and E' are two equivalence classes, then either $E = E'$ or $E \cap E' = \emptyset$.

(iii) Show that

$$\bigcup_{E \in X/\sim} E = X.$$

Exercise 24.6.3. Let X, Y be sets and $f : X \to Y$ a function. Let $\sim$ be the equivalence relation from Example 24.3.3.

Exhibit a bijection between $f(X)$ and $X/\sim$.

Exercise 24.6.4. Let $A \subset X \times X$ be any subset. Show that there exists an *smallest* equivalence relation R_A so that $A \subset R_A$.

More precisely, construct an equivalence relation R_A with $A \subset R_A$ such that for any other equivalence relation R such that $A \subset R$, we have that $R_A \subset R$.

This R_A is called the equivalence relation generated by A.

(Hint: Show that the intersection of equivalence relations is an equivalence relation, and that A is contained in *some* equivalence relation; then let R_A be the intersection of all equivalence relations containing A.)