Lecture 25

More on equivalence relations

25.1 Recollections

Last time we saw the definition of equivalence relations, equivalence classes, and the quotient map. Here is a quick and incomplete review:

1. An equivalence relation is a subset $R \subset X \times X$. That $(x, x') \in R$ is the same thing as saying x is related to x'.

2. Given an equivalence relation R, you can talk about equivalence classes. An equivalence class $E \subset X$ is a set of elements all related to each other[1]. If $x \in E$, then we write $E = [x]$. Note that if $x \sim x'$, we have that $[x] = [x']$.

3. The set $X/\sim$ is the set of all equivalence classes. So $X/\sim$ is a subset of $\mathcal{P}(X)$. This $X/\sim$ is called the quotient set.

4. There is a natural function $X \to X/\sim$ given by $x \mapsto [x]$. This is called the quotient map.

25.2 Partitions

An equivalence relation turns out to be the same thing as a "partition."

[1]Moreover, this set is complete in that if $x \in E$ and if x' is any element related to x, then $x' \in E$. It is also exclusive in that no unrelated elements are in E—so if $x, x' \in E$, then $x \sim x'$. A succinct way to summarize the last two sentences is if $x \in E$, then "$x' \in E \iff x' \sim x$."

A partition of a set X is a way to break up the set X into pieces. More rigorously:

Definition 25.2.1. Let X be a set. A *partition* of X is a collection $P \subset \mathcal{P}(C)$ for which

1. $\forall A \in P$, $A \neq \emptyset$.
2. $(A, B \in P) \wedge (A \neq B) \implies A \cap B = \emptyset$.
3. $\bigcup_{A \in P} A = X$.

In words: A partition P is a collection of subsets of X for which

1. No element of P is empty.
2. The elements of P are disjoint.
3. The union of the elements of P is X itself.

Example 25.2.2. The following are partitions of $X = \{1, 2, 3\}$:

1. $P = \{\{1, 2\}, \{3\}, \}$
2. $P = \{\{1\}, \{2\}, \{3\}\}$
3. $P = \{\{1\}, \{2, 3\}, \}$
4. $P = \{\{1, 3\}, \{2\}, \}$
5. $P = \{\{1, 2, 3\}\}$.

The following are not partitions of X:

1. $P = \{\{1, 2\}\}$
2. $P = \{\{1\}, \{2\}, \{\}\}$
3. $P = \{\{1\}, \{2\}, \{3\}, \{\}\}$
4. $P = \{\{1, 2\}\}$.

Proposition 25.2.3. An equivalence relation on X is the same thing as a partition of X.

More rigorously, there is a natural bijection between the set of all equivalence relations on X, and the set of all partitions on X.

The first statement of the proposition is "the way you should think" about equivalence relations and partitions. The second statement is the rigorous way in which we formulate how these are the "same thing"—the collection of these things admit a natural bijection between them.

Remark 25.2.4. So, even though it is easier to think of the "things"—i.e., the equivalence relation, and the partition—we make the notion that they are equivalent precise by exhibiting a bijection between the collection of all such things. This bijection is what tells us we can "go back and forth" between the two concepts.

Proof. Let *Rel* be the set of all equivalence relations on X. (So this is the set of all $R \subset X \times X$ satisfying the usual conditions—*Rel* is a subset of $\mathcal{P}(X \times X)$.)

Let *Part* be the set of all partitions on X. (*Part* is a subset of $\mathcal{P}(\mathcal{P}(X))$, in case you're keeping score at home.)

Here are two functions:

$$f : Rel \to Part, \qquad R \mapsto X/\sim$$

and

$$g : Part \to Rel \qquad P \mapsto \{(x, x') | \exists A \in P \, (x, x' \in A)\}.$$

It turns out that $f(g(P)) = P$ and $g(f(R)) = R$, so that f and g are mutually inverse. This shows they are each bijections. □

Remark 25.2.5. But in real life, it is often easier to think of an equivalence relation as a way of "declaring things equivalent." Partitions are a nice set-theoretic way to organize a given equivalence relation, but not much more than that.

25.3 Example: Integers

We have created $\mathbb{N}$ using Peano's axioms. I now claim that we can "create" $\mathbb{Z}$ from knowing about $\mathbb{N}$, direct products, addition, and equivalence relations.

Let $X = \mathbb{N} \times \mathbb{N}$. We'll write an element of X as (a, b). For example, $(0, 3)$ and $(3, 0)$ are (different) elements of X.

We declare

$$(a, b) \sim (a', b') \iff a + b' = a' + b. \qquad (25.3.0.1)$$

Exercise 25.3.1. Check that this is an equivalence relation. (This means you have to check symmetry, reflexivity, and transitivity.)

Example 25.3.2. $(5, 2) \sim (3, 0)$ and $(1, 4) \sim (0, 3)$.

This equivalence relation probably seems weird to you. But here's how to think of it: Given $(a, b) \in X$, pretend that (a, b) represents the number $a - b$. Then you see that the relation declares (a, b) and (a', b') equivalence precisely if $a - b = a' - b'$. In particular, $(a, 0)$ represents the number a, and informally, we'll want $(0, a)$ to represent the number "negative a."

Remark 25.3.3. Now, you know how to "pretend" this because you know what subtraction is. But just as we knew what a natural number is, and needed Peano's axioms to make it precise, it turns out that this equivalence relation is one rigorous way to make subtraction, and hence negative numbers, precise.

In fact, I want to claim the following:

Theorem 25.3.4. There is a bijection

$$f : X/\sim \to \mathbb{Z}$$

satisfying the property that

$$f([(a + a', 0)]) = f([a, 0]) + f([a', 0])$$

and

$$f([(a, 0)]) + f([(0, a)]) = f([(0, 0)]).$$

Remark 25.3.5. The fact that f is a bijection says that we can think of $X/\sim$ as "equivalent" in size to $\mathbb{Z}$. Of course, this doesn't say much, because we know $\mathbb{Z}$ is countable.

The next statement is the interesting bit: This bijection actually "respects" addition, and also sends $[(0, a)]$ to a number that acts like the negative of $[(a, 0)]$. This is what makes our previous "pretending" into rigorous math.

First, a small fact that might help us.

Lemma 25.3.6. Every element $(a, b) \in X$ is related to an element of the form $(0, n)$ or $(n, 0)$.

Proof. If $a - b \geq 0$, set $n = a - b$. Then

$$a + 0 = n + b$$

so $(a, b) \sim (n, 0)$ by (25.3.0.1).

If $a - b \leq 0$, set $n = b - a$. Then

$$a + n = 0 + b$$

so $(a, b) \sim (0, n)$ by (25.3.0.1). □

Remark 25.3.7. We can summarize the situation as follows:

- If $a - b \geq 0$, then $(a, b) \sim (a - b, 0)$.
- If $a - b \leq 0$, then $(a, b) \sim (0, b - a)$.

Proof of Theorem 25.3.4. Define f as follows:

$$f([(a, b)]) = a - b.$$

In other words, given an equivalence class $E \in X/\sim$, we choose an element $(a, b) \in E$, and compute the integer $a - b$. This is the integer we assign E to.

We should first make sure that f is "well-defined," which means that $a - b$ doesn't depend on the element (a, b) we chose in E.

So suppose (a', b') is another element in E. We have to check that $a - b = a' - b'$. But this follows straightforwardly from knowing (25.3.0.1).

So now that we have defined f, let's check that it's a bijection.

I first claim f is an injection. To see this, suppose $f([(a, b)]) = f([(a', b')])$. Then we have that $a - b = a' - b'$, from which we see that $a + b' = a' + b$, meaning $(a, b) \sim (a', b')$. So $[(a, b)] = [(a', b',)]$.

Now I claim f is a surjection. Given a positive integer n, we know that $f([(n, 0)]) = n$. Given a negative integer n, we can compute that $f([(0, -n)]) = n$.

We have finished showing f is a bijection.

To check the rest of the theorem, note

$$f([(a + a', 0)]) = a + a' - 0 = (a - 0) + (a' - 0) = f([(a, 0)]) + f([(a', 0)]).$$

A similar proof proves the last statement of the theorem. □

Remark 25.3.8. This proof assumes you know what $\mathbb{Z}$ is, and that you know what subtraction is.

In actuality, the way to *construct* the set $\mathbb{Z}$, and the way to *define* subtraction, is to pass through the set X and this equivalence relation.

In other words, we *define* $\mathbb{Z}$ to be the set $X/\sim$. And we define negation by declaring $-[(a,b)] = [(b,a)]$.