Lecture 26

$\mathbb{R}$ is uncountable. What's next?

Today is the last lecture. We'll prove:

Theorem 26.0.1. There exists a bijection between $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$.

By Cantor's Theorem, we conclude:

Corollary 26.0.2. $\mathbb{R}$ is not countable.

And the proof will use Cantor-Schröder-Bernstein. So we'll be using every major idea we've learned in this class!

26.1 Some Lemmas

Let's leave these lemmas as black boxes; we'll prove them later in the notes.

Lemma 26.1.1. There exists a bijection between $\mathcal{P}(\mathbb{N})$ and the interval $[0, 2]$.

Lemma 26.1.2. $\mathbb{R}$ admits a bijection to the open interval $(0, 1)$.

And for good measure, let me remind you of the following:

Theorem 26.1.3 (Cantor-Schröder-Bernstein, Theorem 18.1.1)**.** If there exists an injection from X to Y, and an injection from Y to X, then there exists a bijection from X to Y.

26.2 The proof

Proof of Theorem 26.0.1. First, there is a bijection $\mathcal{P}(\mathbb{N}) \cong [0,2]$ by Lemma 20.5.

So it suffices to show that there exists a bijection between $[0,2]$ and $\mathbb{R}$. We will use Cantor-Schöder-Bernstein to prove this. In other words, all we need to show is that there is an injection from $[0,2]$ to $\mathbb{R}$, and an injection from $\mathbb{R}$ to $[0,2]$.

(1) It is obvious there is an injection from $[0,2]$ to $\mathbb{R}$ because $[0,2] \subset \mathbb{R}$.

(2) By Lemma 26.1.2, there is a bijection

$$\mathbb{R} \xrightarrow[26.1.2]{\cong} (0,1).$$

There is obviously an injection

$$(0,1) \hookrightarrow [0,2]$$

because $(0,1) \subset [0,2]$.

So the composition gives an injection $\mathbb{R} \hookrightarrow [0,2]$.

This completes the proof. □

26.3 What's next?

Before we prove the lemmas, I want to take a moment to discuss where you can go next in mathematics.

26.3.1 Mysteries of the logical universes

We've seen that there is no such thing as a largest set; and in particular, there is no such thing as a set of all sets. There is a *different* paradox, called Russell's paradox, which tells us why we should not be able to even discuss "the set of all sets" in logic; more accurately, if you think of logical statements as statements that can be built out of more basic statements, "set of all sets" is actually a phrase that should not be allowed to be built. (This is a *different* reason, independent of size, as to why the set of all sets should not exist.)

Depending on your taste, this is either mumbo-jumbo, or incredibly deep and interesting. If you are interesting in understanding more about formal logic; how thinking beings can create and define the rules of a logical universe; and what mathematical results rely on such rules; the person to talk to in our department is Will Boney.

26.3.2 Shapes and Topology

More than once, you probably wanted to draw pictures in this class. For example, $[0, 2]$ is not easily thought of as a set, but easily drawn as a closed interval.

The ability to draw a set has hidden within it more structures of that set—we think of the elements of $[0, 2]$ as ordered (we know when one is bigger than another) and in some sense "without gaps." It has a *shape*.

Topology is the study of shapes. Though 4330 is a bit abstract and often fails to explain why the things you learn there have anything to do with shapes, a class like 4337D will allow you to explore these things. You'll learn how to use the ideas from this class as a foundation to talk about more abstract shapes: In higher dimensions, in real life, in data sets.

26.3.3 Symmetries and modern algebraic notions

Modern algebra, or abstract algebra, gives powerful ways to think about symmetries. It was kicked off by a mathematician named Evariste Galois, and also Sophus Lie; thanks to them, it turns out that solutions to the equation $x^8 - 2x^4 - 4$ has *exactly* the symmetries of a square. Don't worry, this sentence didn't make sense to most mathematicians before the 1900's, either. But you can see how powerful we can become if we realize that we can think about symmetries of solutions to equations the same way we can think about symmetries of shapes.

26.3.4 What is $\mathbb{R}$? Why does calculus make sense?

We got to talk about how to articulate the existence of $\mathbb{N}$ (Peano's axioms), how to construct $\mathbb{Z}$ and $\mathbb{Q}$. We haven't talked about how to construct $\mathbb{R}$, which really is the beginning of a field called "analysis." And once you construct $\mathbb{R}$, you can finally begin to dot the Is and cross the Ts of calculus—why did all that stuff even function properly?

26.4 Proof of Lemma 26.1.1

You know about decimal expansions of a real number. This proof uses the idea of a "binary expansion."

In a decimal expansion, we insert one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 at every place to express a number:

$$13.267\ldots.$$

And, to be precise, the above decimal represents the number

$$(1 \times 10^1) + (3 \times 10^0) + (2 \times 10^{-1}) + (6 \times 10^{-2}) + (7 \times 10^{-3}) + \ldots.$$

This is called a "base 10" expansion.[1] In computer science, and sometimes n math, "base 2" expansion is more common. Just as base 10 is called decimal, sometimes base 2 is called binary.

In base 2, you are only allowed to use the digits 0 and 1. So a number might look like

$$1101.01101\ldots$$

and the above would represent the number

$$(1\times2^3)+(1\times2^2)+(0\times2^1)+(1\times2^0)+(0\times2^{-1})+(1\times2^{-2})+(1\times2^{-3})+(0\times2^{-4})+(1\times2^{-5})+\ldots.$$

And every real number can be expressed this way.

Now, what does this have to do with establishing a bijection between $\mathcal{P}(\mathbb{N})$ and $[0,2]$?

Well, first, we get an injection from $[0,2]$ to $\mathcal{P}(\mathbb{N})$ by doing the following. Given a real number x, write out a binary expansion. If $x \in [0,2]$, we can write a binary expansion where the whole number's place (i.e., the 0th place) is either 0 or 1. Then one can construct a set $g(x) \subset \mathbb{N}$ so that $n \in g(x)$ exactly if the $-n$th place of the expansion is 1.

Example 26.4.1. If x has binary expansion

$$1.1000100000\ldots,$$

notice that the 0th place, the 1st place, and the 5th place are equal to 1, and all other places are 0. $Sog(x) = \{0,1,5\}$.

Here are more examples:

(a) If $x = 0.000000\ldots$, $g(x) = \emptyset$.

[1] We probably use base 10 because we have 10 fingers, though I have no scientific basis for this claim.

(b) If $x = 0.10000000\ldots$, $g(x) = \{1\}$.

(c) If $x = 0.1010100000\ldots$, $g(x) = \{1, 3, 5\}$.

(d) If $x = 1.11111\ldots$, $g(x) = \mathbb{N}$.

(e) If $x = 0.11111\ldots$, $g(x) = \mathbb{N} \setminus \{0\}$.

This defines a function $g : [0, 2] \to \mathcal{P}(\mathbb{N})$, and g is an injection—for if $g(x) = g(x')$, then x and x' has the same binary expansion, so they are the same number.

On the other hand, we can define a function $f : \mathcal{P}(\mathbb{N}) \to [0, 2]$ that is also an injection. Given a subset $A \subset \mathbb{N}$, declare $f(A)$ to be the binary expansion all of whose odd places are given by 0, but whose $-2n$th place is given by 0 if $n \notin A$ and 1 if $n \in A$.

Example 26.4.2. 1. If $A = \mathbb{N}$, then $f(A) = 0.101010101010\ldots$.

2. If $A = \{1, 2, 3\}$, then $f(A) = 0.101010000000\ldots$.

3. If $A == \emptyset$, then $f(A) = 0.0000000\ldots$.

Then f is an injection.[2]

Proof of Lemma 26.1.1. We have shown that $\mathcal{P}(\mathbb{N})$ and $[0, 2]$ admits injections to each other (called f and g above); so by Cantor-Schröder-Bernstein, there is some bijection between them. $\square$

26.5 Proof of Lemma 26.1.2

Proof of Lemma 26.1.2. The function $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection because it has an inverse called arctan. So we have a bijection as follows:

$$\mathbb{R} \xrightarrow[\arctan]{\cong} (-\pi/2, \pi/2).$$

Now, the operation $x \mapsto \frac{1}{\pi}x$, otherwise known as scaling by a factor of $\frac{1}{\pi}$, is a bijection

$$(-\pi/2, \pi/2) \xrightarrow[x \mapsto x/\pi]{\cong} (-1/2, 1/2).$$

[2]This would not be true if we declared $f(A)$ to be the number whose $-n$th place is given by 1 if $n \in A$, and 0 otherwise. The problem is that the number $1.00000\ldots$, given by $A = \{0\}$, would equal the number $0.111\ldots$, given by $A = \mathbb{N} \setminus \{0\}$.

(To see this, note there is an inverse called multiplication by π.)

Finally, the operation of $+\frac{1}{2}$ gives a bijection[3] as follows:

$$(-1/2, 1/2) \xrightarrow[x \mapsto x+\frac{1}{2}]{\cong} (0,1).$$

A composition of bijections is a bijection; so composing the above functions, we obtain the desired result. $\square$

[3]This function has an inverse given by $x \mapsto x - \frac{1}{2}$, for example.