Lecture 1

What is (abstract) algebra?

1.1 Goals

1. Begin to think about the operations we take for granted, or utilize, in our usual number system.

2. Be exposed to the idea that operations can exist for other kinds of "number systems."

3. Explore what properties of the operations we perform arise are fundamental, and which properties can be derived from the fundamental ones.

1.2

When you think of algebra, you probably think of what most of the population thinks of: Something involving the symbol x, or something involving manipulation of polynomials. Maybe you think about a sequence of equations like this:

$$\begin{aligned} x^2+3x-5 &= 7x-9 \qquad (1.2.0.1)\\ x^2+3x+4 &= 7x \\ x^2-4x+4 &= 0 \\ (x-2)^2 &= 0 \\ \implies x-2 &= 0 \\ x &= 2 \end{aligned}$$

Confusingly, the sense in which "algebra" is used by modern researchers in pure math (and the sense in which we will use it in this course) is a little different. Let me try to articulate the difference as follows:

- *Algebra* (as most humans understand it) is a *tool* for solving problems by *manipulating* operations.
- *Abstract algebra*, or just "algebra" (as pure math researchers understand it) is the *study/science* of objects that *have* operations.

This description probably makes no sense, because you haven't knowingly encountered many objects that "have" operations of the sort you're used to. Ignoring that for the moment, you can see the power of algebra (in the sense of modern pure math research): The sequence of operations that we see in (1.2.0.1) makes sense in many other contexts—not just in the usual setting of solving for a "number" x. Put another way, algebra-the-science is the study of all the settings in which algebra-the-tool may be used.

Example 1.2.1. Let x be a two-by-two matrix. We'll write out its entries:

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, for example, the operation $3x$ makes sense:

$$3x = \begin{pmatrix} 3a & 3b \\ 3c & 3d \end{pmatrix}.$$

Interpreting 5 as 5 times the identity matrix[1], $3x-5$ also makes sense:

$$3x-5 = \begin{pmatrix} 3a-5 & 3b \\ 3c & 3d-5 \end{pmatrix}.$$

[1]We might see why we would do have such an interpretation later in the course

Now, if x is a matrix such that (1.2.0.1) holds, we can do identical work to arrive at the equation

$$(x-2)^2 = 0.$$

However, one amazing property of matrices is that a matrix can square to 0 without being the 0 matrix. For example, the square of the following matrix is zero[2]:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

So, unlike in the ordinary system of numbers, there is not *one* solution to the equation $(x-2)^2 = 0$. There are infinitely many when x is a 2-by-2 matrix. All of the following are examples of x that satisfy the equation $(x-2)^2 = 0$.

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 0 \\ \pi & 2 \end{pmatrix}.$$

Remark 1.2.2. As with any example in a textbook, or in lecture notes, you should make sure you understand the above example—right now. For example, do you remember how to add matrices? How to multiply two matrices (and hence, how to square a single matrix)? Can you verify all the claims I made in the above example? This is a good habit to get into. If you cannot comfortably verify the claims, it's a great signal that you should practice, or that you should go to Office Hours, or that you should write the professor an e-mail with questions.

The take-away that I want you to have from the above example is (i) The algebra you've learned so far is more powerful than you even knew: It can apply to many contexts where you're not just "solving for (a number) x," and (ii) There are interesting, and new, phenomena to be discovered by studying other settings in which the operations of "usual algebra" make sense. (How cool is that *infinitely many* things can square to zero?)

1.3 Operations

Now it's time for class discussion. If you're reading these notes but cannot come to class, you should explore the following questions and write your thoughts down. Make sure you spend a good amount of time thinking about

[2]You should check!

the questions; I'll give at least 30 minutes in class to answer these questions in groups.

1. What "operations" do you know how to do on numbers? Here are some more specific questions that may help you explore:

   (a) There are operations that take in a single number, and output another. For example, given a number a, you can ask what the cube root of a is. Can you think of more of these operations? Are there operations that feel more "basic" then, say, taking nth roots?

   (b) There are operations that take in two numbers, and output another. For example, given two numbers a and b, we can produce a number called $a+b$. Can you think of more such *binary* operations?

2. Are there any special relationships that these operations have? For example, is there a relationship between $\sqrt[3]{a+b}$ and $\sqrt[3]{a}+\sqrt[3]{b}$? How about among the binary operations you thought of?

3. Are there any special numbers that behave very nicely with respect to the operations you came up with?

1.4 A note to educators

"Abstract algebra" or "modern algebra" has, broadly speaking, two very distinct subfields: (I) The subfield involving rings, where addition and multiplication are omnipresent, and (II) The subfield involving groups, where symmetry is the name of the game.

It is a source of constant confusion that most abstract algebra classes taught by pure mathematicians seem to teach (II), but (II) is the subfield with the *least* obvious connection to high school and college algebra! So many students leave their abstract algebra classes wondering, "how was that *algebra*?"

The approach of this course is to begin with (I), where the presence of addition and multiplication, along with the prominent use of polynomials, really makes the curriculum feel like it builds on what students already conceive of as "algebra." Theorem 6.6.4 is a great example of how high-school-algebra questions can be asked in new settings, and how abstract-algebra techniques can answer these questions simply by boot-strapping high-school-algebra knowledge.

1.5 Your homework for next time

For homework, I want you to go to our Canvas website and respond to the Discussion prompt I posted. The prompt should read as follows:

1. What operations can we perform on real numbers? (You can speak about operations that output one number from a given number, or binary operations which input two numbers and output one number.)

2. Of the operations that you've thought about, which ones feel most "fundamental" to you, personally?

3. Which of the operations have a definition that depends on knowing other operations?