Lecture 3

Matrices

3.1 Goals

1. Review summation notation as needed
2. Get used to matrix addition and multiplication
3. Get used to proofs using matrices

3.2 Summation notation review

You've probably seen summation notation previously. They typically come up when performing Riemann sums in Calculus.

Mathematicians use *summation notation*, or *sigma notation* to efficiently describe the addition of many numbers. I've also told you that there are rings that consist of things that aren't just numbers—in fact, we will use summation notation to describe addition in any ring.

Definition 3.2.1. Let $a_1, \ldots, a_n$ denote a collection of numbers (or, more generally, elements of a ring). Then we let

$$\sum_{k=1}^{n} a_k$$

denote the summation $a_1 + a_2 + \ldots + a_k$. In fact, choosing two numbers i

and j so that $1 \leq i \leq j \leq k$, the notation

$$\sum_{k=i}^{j} a_k$$

denotes the summation $a_i + a_{i+1} + \ldots + a_{j-1} + a_j$.

Example 3.2.2. Let us denote a collection of numbers as follows:

$$a_1 = 3, \qquad a_2 = 5, \qquad a_3 = 5, \qquad a_4 = 5, \qquad a_5 = -2, \qquad a_6 = \frac{1}{2}, \qquad a_7 = \pi.$$

Then

$$\sum_{k=1}^{7} a_k = 3 + 5 + 5 + 5 - 2 + \frac{1}{2} + \pi = \pi + 16.5 = \pi + \frac{33}{2}.$$

And

$$\sum_{k=3}^{6} a_k = 5 + 5 - 2 + \frac{1}{2} = \frac{17}{2}.$$

3.3 Matrices

Recall that an *m-by-n matrix* is an array of numbers containing m rows and n columns. Here are some examples:

$$\begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix}, \qquad \begin{pmatrix} \pi & -3 \\ \sqrt{2} & 4 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 5 & 5 & 7 & -3 \\ -8 & 5 & 0 & \frac{1}{2} \end{pmatrix}.$$

The above are 2-by-2, 3-by-2, and 2-by-4 matrices, respectively.

Notation 3.3.1. Let A be an m-by-n matrix. We let $A_{i,j}$ denote the entry in the ith row and jth column.

Example 3.3.2. In the matrix

$$B = \begin{pmatrix} -2 & \sqrt{2} \\ 1 & \pi \\ \frac{2}{3} & 3 \end{pmatrix}$$

we see that

$$B_{1,1} = -2, \qquad B_{1,2} = \sqrt{2}, \qquad B_{2,1} = 1, \qquad B_{2,2} = \pi, \qquad B_{3,1} = \frac{2}{3}, \qquad B_{3,2} = 3.$$

Definition 3.3.3 (Equality of matrices)**.** We say that two m-by-n matrices A and B are equal, and write $A = B$, if their entries agree. That is, we write $A = B$ if for every i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$, we have

$$A_{i,j} = B_{i,j}.$$

If you have an m-by-n matrix A, and a k-by-l matrix B, you can multiply the two matrices to obtain a new matrix AB if $n = k$.

Definition 3.3.4 (Matrix multiplication)**.** Let A be an m-by-n matrix and B a n-by-l matrix. (Note the number of columns of A equals the number of rows of B.) define the product AB to be an m-by-l matrix, whose entries are given by

$$(AB)_{i,j} = \sum_{k=1}^{n} A_{i,k}B_{k,j}.$$

Example 3.3.5.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & -2 & -3 \\ 0 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 2 & -1 \\ 0 & 9 & 2 & -5 \\ 0 & 13 & 2 & -9 \end{pmatrix}.$$

For example, the $(1,4)$ entry of the result is obtained by the summation

$$(1)(-3) + 2(1) = -3 + 2 = -1.$$

If you like, this is the dot product between the first row of the left matrix, and the 4th column of the second matrix.

Definition 3.3.6 (Matrix addition)**.** Let A and B be two m-by-n matrices. Then their sum is defined as another m-by-n matrix whose (i,j)th entry is given by

$$(A + B)_{i,j} = A_{i,j} + B_{i,j}.$$

Example 3.3.7.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} \pi & 2 & 3 \\ 6 & 7 & 1 \end{pmatrix} = \begin{pmatrix} 1+\pi & 4 & 6 \\ 10 & 12 & 7 \end{pmatrix}$$

Notation 3.3.8. Fix an integer $n \geq 1$. We let

$$M_n(\mathbb{R})$$

denote the set of all n-by-n matrices whose entries are real numbers.

More generally, let R be any ring. Then $M_n(R)$—the set of all matrices with entries in R—is a ring.

Remark 3.3.9. So if $A \in M_n(R)$, then for any i, j with $1 \le i \le n$ and $1 \le j \le n$, we have that $A_{i,j} \in R$.

3.3.1 Example of a proof involving matrices

Let's see an example of a proof utilizing matrices.

Definition 3.3.10. Fix an integer $n \ge 1$ and let A be an n-by-n matrix. We say that A is *lower-triangular* if $i > j \implies A_{i,j} = 0$.

Proposition 3.3.11. If A and B are both lower-triangular, then AB is also lower-triangular.

Proof. We must show that if $i > j$, then $(AB)_{i,j} = 0$. So let us suppose that $i > j$. We of course have

$$(AB)_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{k,j} \qquad \text{Definition of matrix multiplication.}$$

We can split up this summation as follows:

$$\begin{aligned} \sum_{k=1}^{n} A_{i,k} B_{k,j} &= \left(\sum_{k=j+1}^{n} A_{i,k} B_{k,j} \right) + \left(\sum_{k=1}^{j} A_{i,k} B_{k,j} \right) \\ &= \left(\sum_{k=j+1}^{n} A_{i,k} \cdot 0 \right) + \left(\sum_{k=1}^{j} 0 \cdot B_{k,j} \right) \\ &= 0 + 0. \end{aligned} \tag{3.3.1.1}$$

Note that when $j+1 \le k \le n$, we know $k > j$, so $B_{k,j} = 0$ by the assumption that B is lower-triangular. Likewise, when $1 \le k \le j$, we know that $i > k$ because we are computing $(AB)_{i,j}$ where $i > j$; so $A_{i,k} = 0$ in this range of k. $\square$

3.4 Exercises on matrix addition and multiplication

Exercise 3.4.1. Compute the following operations. If the operation do not make sense as stated, answer "does not make sense."

(a) $\begin{pmatrix} 3 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 1 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Exercise 3.4.2. Compute the following operations. If the operation do not make sense as stated, answer "does not make sense."

(a) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 5 & 6 \\ 2 & 4 & 2 \end{pmatrix}$

Exercise 3.4.3. Compute the following operations. If the operation do not make sense as stated, answer "does not make sense."

(a) $\begin{pmatrix} 3 & 5 & 6 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} \pi & a & b \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 5 & 6 \\ 2 & 4 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 5 & 6 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} \pi & a & b \\ 1 & 3 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$

Exercise 3.4.4. Compute the following operations. If the operation do not make sense as stated, answer "does not make sense."

(a) $\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & -1 \\ 0.2 & 3 \\ 4 & 5 \\ 6 & 7 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0.2 & 3 \\ 4 & 5 \\ 6 & 7 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

Exercise 3.4.5. Compute the following operations. If the operation does not make sense as stated, answer "does not make sense."

(a) $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Exercise 3.4.6. Fix an integer $n \geq 1$ and a real number λ.[1] Suppose that A is an n-by-n matrix with entries as follows:

$$A_{i,j} = \begin{cases} 0 & i \neq j \\ \lambda & i = j. \end{cases}$$

Such a matrix is called a *diagonal* matrix.

(a) Show that if B is an arbitrary n-by-n matrix, then $AB = BA$.

(b) How does the (i, j)th entry of AB compare to the (i, j)th entry of B?

Exercise 3.4.7. Let A be an n-by-n matrix. A is called *upper-triangular* if $i > j \implies A_{i,j} = 0$.

(a) Show that if A and B are two upper-triangular matrices, then so is AB.

(b) Show that if A and B are two upper-triangular matrices, then so is $A+B$.

Exercise 3.4.8. Let A be an n-by-n matrix. The *transpose* of A is a matrix whose (i, j)th entry is the (j, i)th entry of A. We let A^T denote the transpose. Show that $(AB)^T = B^T A^T$.

---

[1] This is the Greek letter lambda. It is pronounced "lam-da."

3.5 The ring of n-by-n matrices

Fix $n \geq 1$. Given two n-by-n matrices, we know how to add them (Definition 3.3.6) and we know how to multiply them (Definition 3.3.4). So it's beginning to look like the set of all n-by-n-matrices might form a ring!

We have the following, which we'll call a theorem mainly because it is rather tedious to dot all the is and cross all the ts.

Theorem 3.5.1. Fix a ring R and any integer $n \geq 1$. Then $M_n(R)$, equipped with the addition and multiplication of matrices, is a ring.

The proof of this theorem is rather tedious, so you should only read the parts you care about. Even if you don't read the proof, you should just know that the collection of all n-by-n matrices, with entries in a given ring R, forms a ring itself.

Proof that addition is associative. We have that

$$\begin{aligned}(A+(B+C))_{i,j} = A_{i,j} + (B+C)_{i,j} &= A_{i,j} + (B_{i,j} + C_{i,j}) \\ &= (A_{i,j} + B_{i,j}) + C_{i,j} \\ &= (A+B)_{i,j} + C_{i,j} \\ &= ((A+B)+C)_{i,j}\end{aligned}$$

where every equality except the second equality is by definition of matrix addition. The second equality uses the fact that R is a ring (so that addition in R is associative).

We see that the entries of $A + (B + C)$ and $(A + B) + C$ agree, so the two matrices are equal. □

Proof that addition is commutative. We have that

$$\begin{aligned}(A+B)_{i,j} &= A_{i,j} + B_{i,j} \\ &= B_{i,j} + A_{i,j} \\ &= (B+A)_{i,j}.\end{aligned}$$

where the first and third equality follow from the definition of matrix addition, and the second equality is by the fact that R is a ring (so that addition is commutative). This shows $A + B = B + A$. □

Proof that there is an additive identity. Let 0 be the additive identity of the ring R, and let $\underline{0}$ be the n-by-n matrix all of whose entries are 0. Then

$$(A+\underline{0})_{i,j} = A_{i,j} + \underline{0}_{i,j} = A_{i,j} + 0 = A_{i,j}.$$

Thus, $A+\underline{0}$ and A are the same matrix. $\square$

Proof that additive inverses exist. Given an n-by-n matrix A, let $-A$ denote the matrix whose (i,j)th entry is given by $-(A_{i,j})$. (That is, by the additive inverse of $A_{i,j}$ in the ring R.) Then

$$(A+(-A))_{i,j} = A_{i,j} + (-A)_{i,j} = A_{i,j} + -(A_{i,j}) = 0$$

Hence $A + -A = \underline{0}$. $\square$

Proof that multiplication is associative. We have

$$\begin{aligned}
(A(BC))_{i,j} &= \sum_{k=1}^{n} A_{i,k}(BC)_{k,j} && \text{Definition of matrix multiplication} \\
&= \sum_{k=1}^{n} A_{i,k}\left(\sum_{l=1}^{n} B_{k,l}C_{l,j}\right) && \text{Definition of matrix multiplication} \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} A_{i,k}(B_{k,l}C_{l,j}) && \text{Distributivity in } R \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} (A_{i,k}B_{k,l})C_{l,j} && \text{Associativity of multiplication in } R \\
&= \sum_{l=1}^{n}\sum_{k=1}^{n} (A_{i,k}B_{k,l})C_{l,j} && \text{Addition is commutative} \\
&= \sum_{l=1}^{n}\left(\sum_{k=1}^{n} A_{i,k}B_{k,l}\right)C_{l,j} && \text{Distributivity} \\
&= \sum_{l=1}^{n} (AB)_{i,l}C_{l,j} && \text{Definition of matrix multiplication} \\
&= ((AB)C)_{i,j} && \text{Definition of matrix multiplication.}
\end{aligned}$$

This shows $A(BC) = (AB)C$. $\square$

Proof that there is a multiplicative identity. Let 1 be the multiplicative identity of R, and 0 the additive identity. Let I denote the n-by-n matrix for

which
$$I_{i,j} = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$
We claim that I is a multiplicative identity in $M_n(R)$. To see this, note
$$\begin{aligned} (AI)_{i,j} &= \sum_{k=1}^{n} A_{i,k} I_{k,j} \\ &= A_{i,j} I_{j,j} + \sum_{k \neq j} A_{i,k} 0 \\ &= A_{i,j}. \end{aligned}$$
This shows $AI = A$. Similar work shows $IA = A$. □

Proof that multiplication is distributive.
$$\begin{aligned} (A(B+C))_{i,j} &= \sum_{k=1}^{n} A_{i,k}(B+C)_{k,j} \\ &= \sum_{k=1}^{n} A_{i,k}(B_{k,j} + C_{k,j}) \\ &= \sum_{k=1}^{n} (A_{i,k}B_{k,j} + A_{i,k}C_{k,j}) \\ &= \sum_{k=1}^{n} A_{i,k}B_{k,j} + \sum_{k=1}^{n} A_{i,k}C_{k,j} \\ &= (AB)_{i,j} + (AC)_{i,j}. \end{aligned}$$
This shows that $A(B+C) = AB + AC$. Note that the third equality uses that R is a ring (hence that multiplication distributes over addition in R). The fourth equality is parsing summation notation, and all other equalities are just definitions of matrix operations.

A similar proof shows that $(B+C)A = BA + CA$. □

3.6 Some remarks

It turns out that matrices are by far among the most useful tools in modern mathematics. It's useful not just in pure mathematics, but also in many fields of applied mathematics. The person who came up with the word "matrix" is James Joseph Sylvester—this happened back in the 1850's. But matrices in some form were used by Babylonians, and also during the Han dynasty of China, sometime between 300 BCE and 200 CE.

(By the way, a good way to test for whether an idea is important, or natural, is to see if people in far-away places came up with the idea independently.)

All the proofs you performed in the exercises are valid over any ring. You may have had in mind real-valued matrices, but in fact there is something very "universal" about the proofs, in that they are true not just in $M_n(\mathbb{R})$ but also in $M_n(R)$ for R an arbitrary ring.

3.7 Extra credit: Centers

Let R be a ring. The *center* of a ring is the set of all elements $a \in R$ so that, for every $b \in R$, $ab = ba$. We write $Z(R)$ for the center of a ring.

1. Prove that $Z(R)$ is a ring.

2. Tell me what $Z(M_2(\mathbb{R}))$ is.