Lecture 5

Complex numbers

5.1 Goals

1. Recall how to add and multiply complex numbers
2. Understand that addition of complex numbers is addition of vectors
3. Understand that multiplication by complex numbers is like scaling and rotating
4. Become proficient with computations using complex numbers

5.2 Review of complex numbers

Recall that mathematicians use the symbol i to denote a square root of -1. I say "a" square root and not "the" square root because you always expect two square roots to any non-zero number—and indeed, $-i$ is another square root of -1. We'll see that this ambiguity is a symmetry later in the course.

Definition 5.2.1. A *complex number* is a number obtained by adding a real multiple of i to some real number.

Notation 5.2.2. Thus, any complex number can be written in the form $a + bi$ where $a, b \in \mathbb{R}$.

The set of all complex numbers is denoted by $\mathbb{C}$, which is the blackboard bold font of the capital letter C.

Example 5.2.3. The following are all examples of complex numbers:

$$3, \qquad 3+i, \qquad 3-i, \qquad 3+\sqrt{\pi}i, \qquad i, \qquad \pi+i\pi, \qquad -7i.$$

Note that any real number is a complex number (of the form $b=0$). So we can think of $\mathbb{R}$ as a subset of $\mathbb{C}$.

5.2.1 Addition and multiplication

We can add two complex numbers as follows:

$$(a+bi)+(a'+b'i)=(a+a')+(b+b')i.$$

We can also multiply two complex numbers:

$$(a+bi)(a'+b'i)=aa'+bb'(i)^2+ab'i+a'bi=(aa'-bb')+(ab'+a'b)i. \quad (5.2.1.1)$$

Because I think you are already familiar with complex numbers, I will state the following theorem without proof:

Theorem 5.2.4. $\mathbb{C}$, with the above notions of addition and multiplication, is a commutative ring.

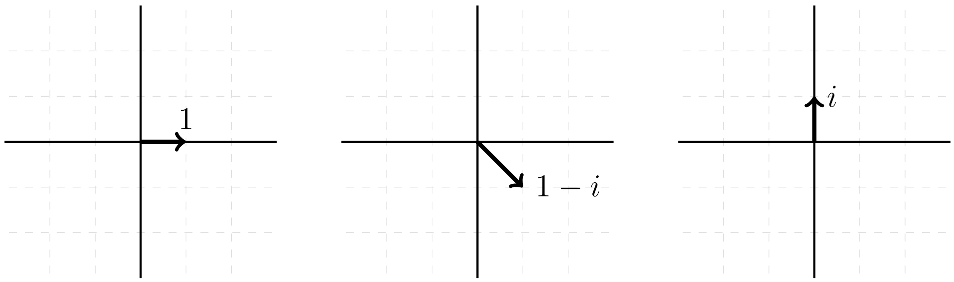
Verifications are left as exercises. However, let me make some things explicit:

- The multiplicative identity is the element $1+0i$, which we simply write as 1.
- The additive identity is the element $0+0i$, which we simply write as 0.

5.2.2 Visualizing complex numbers

To specify a complex number requires two real numbers. Indeed, we see that the sets $\mathbb{C}$ and $\mathbb{R}^2$ admit natural bijections between them: Send the complex number $a+bi$ to the element $(a,b)\in\mathbb{R}^2$. (In the reverse direction, send (x_1,x_2) to the complex number x_1+x_2i.)

So it is customary to visualize a complex number as living in $\mathbb{R}^2$.



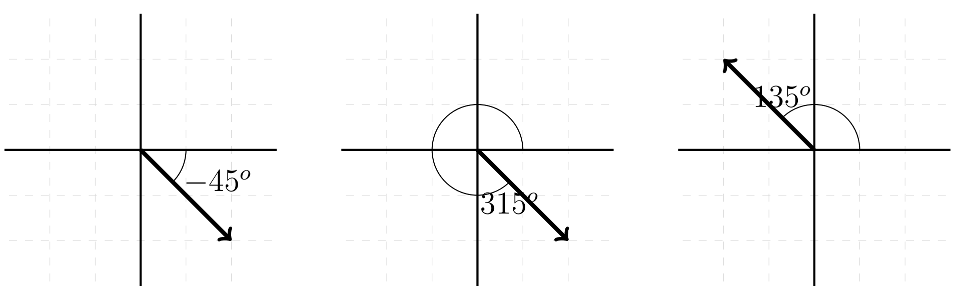
Explicitly, the complex number $a + bi$ is visualized as the point (a, b) in $\mathbb{R}^2$. Thus, one can think of the line with imaginary coordinate 0 (i.e., the line of points with $b = 0$) as a copy of $\mathbb{R}$ sitting "inside" $\mathbb{C}$.

Remark 5.2.5 (Adding complex numbers is adding two-dimensional vectors)**.** By definition of addition, and using this visualization, it is straightforward to check that the addition of complex numbers is visualized as the addition of vectors.

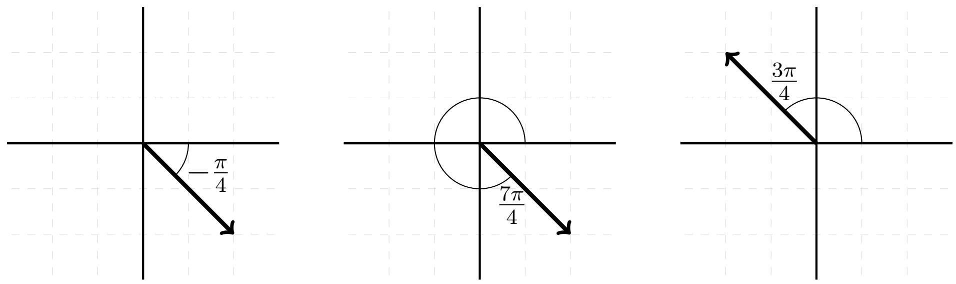
Visualizing complex multiplication requires different reasoning; for this, it helps to recall some facts about polar coordinates and complex numbers.

5.2.3 Polar coordinates

Another way to specify a point in $\mathbb{R}^2$ is to say how far the point is from the origin, and what angle (the vector pointing to) the point makes with the positive x-axis.



Pure mathematicians prefer to use radians, rather than degrees.[1] The above angles in radians are as follows:



Notation 5.2.6. It is customary to indicate the distance from the origin using the variable r, and the angle from the positive x-axis using the variable θ. A complex number of the form $a + bi$ will have distance from the origin given by

$$r = \sqrt{a^2 + b^2}$$

and angle from the positive x-axis given by

$$\theta = \arctan(b/a).$$

5.2.4 Polar coordinates and exponentiation

Polar coordinates take on a deeper meaning when we realize that the function $x \mapsto e^x$ makes sense even if x is a complex number. You may have learned the following as a definition[2] in pre-calculus:

$$e^{a+bi} = e^a(\cos(b) + i\sin(b)). \tag{5.2.4.1}$$

In other words, the image of the exponential function is a lot easier to compute when you use polar coordinates. e^{a+bi} is a complex number with distance e^a from the origin, and whose angle from the positive x-axis is given by b radians. In other words, if you know the polar coordinates r and θ of a non-zero complex number z, we can write

$$z = e^{\ln r + i\theta}.$$

[1] A big reason for doing this is to ensure that the derivative of sin is cos, by the way.

[2] This is fine as a definition, but this definition arises from an amazing fact about power series. See Exercise 5.5.7

When $z = 0$, there are some minor complications (owing to the fact that polar coordinates is not a bijection). z may be represented in polar coordinates by $r = 0$, and by *any* choice of θ. And, z cannot be written in exponential form, as 0 is not in the image of the function e^{blah}.

Recall that when x and x' are real numbers, we know that $e^{x+x'} = e^x e^{x'}$. The following proposition says same is true for complex numbers, with a wonderful consequence:

Proposition 5.2.7. (a) Let z and z' be complex numbers. Then $e^{z+z'} = e^z e^{z'}$.

(b) In particular, suppose that w is a complex number with polar coordinates (r, θ), and w' is a complex number with polar coordinates (r', θ'). Then the product ww' has polar coordinates $(rr', \theta + \theta')$.

In other words, multiplication of complex numbers can be computed by multiplying lengths, and adding angles.

Proof. Proof of (a): Suppose that $z = a + bi$ and $z' = a' + b'i$. Then $e^z = e^a(\cos(b) + i\sin(b))$ by definition of exponentiation for complex numbers. We then see that

$$\begin{aligned}
e^z e^{z'} &= e^a(\cos(b) + i\sin(b)) \cdot e^{a'}(\cos(b') + i\sin(b')) \\
&= e^{a+a'}(\cos(b) + i\sin(b)) \cdot (\cos(b') + i\sin(b')) \\
&= e^{a+a'}[(\cos(b)\cos(b') - \sin(b)\sin(b') + i(\sin(b)\cos(b') + \cos(b)\sin(b'))] \\
&= e^{a+a'}[\cos(b + b') + i(\sin(b + b'))] \\
&= e^{(a+a')+i(b+b')}. \\
&= e^{z+z'}. \qquad (5.2.4.2)
\end{aligned}$$

The next-to-last line is by the definition of exponentiation for complex numbers. Previous to that is the angle addition formula for sine and cosine. All other lines are derived using algebra and the fact that $e^{a+a'} = e^a e^{a'}$.

To see (b), note that the statement is obviously true when w or w' equal zero. (For example if $w = 0$, then $r = 0$, so $rr' = 0$ as well.) So we may as well assume that both w and w' are not zero—in which case $w = e^{\ln r + i\theta}$ where r is the length of w, and θ is the angle w makes with the positive x-axis. Likewise, $w' = e^{\ln(r') + i\theta'}$. Then, by part (a), we know $ww' = e^{(\ln(r) + \ln(r')) + i(\theta + \theta')} = e^{\ln(rr') + i(\theta + \theta')}$, meaning the length of ww' is rr', while the angle is given by $\theta + \theta'$. □

5.3 $\mathbb{C}$ acts on itself (rotation and scaling)

Let us interpret the geometry of Proposition 5.2.7. For this, let z be a complex number. We will think of z as defining a function $\mathbb{C} \to \mathbb{C}$, as follows:

$$\mathbb{C} \to \mathbb{C}, \qquad w \mapsto zw.$$

In other words, z acts on $\mathbb{C}$ by multiplication. What does this do to an input w? Using polar coordinates, we know that z is some complex number with length r and angle θ from the positive x-axis. According to Proposition 5.2.7, if w has length r' and angle θ', then the product zw has length rr' and angle $\theta + \theta'$. Interpreting this geometrically, we see:

1. Multiplication by z *scales* the length of w by a factor of r. (It takes something of length r' to something of length rr'.)

2. Multiplication by z *rotates* w by θ radians. (It takes something of angle θ' from the positive x-axis to something of angle $\theta + \theta'$.)

Thus, multiplication by a complex number has the effect of scaling and rotating.

Remark 5.3.1. Note that the definition of multiplication of complex numbers—(5.2.1.1)—is very algebraic and computable, but does *not*, in any obvious way, give rise to the geometric interpretation of scaling and rotating. We utilized the fact that exponentiation (amazingly) builds a bridge between the algebraic and the trigonometric to see this conclusion.

5.4 Some historical remarks

In school, we learn about complex numbers *way* after we learn about negative numbers. Historically, they arose at around the same time! Gerolamo Cardano, in the 1500's, is typically acknowledged as the first mathematician to make systematic use of negative numbers. (Indeed, it used to be common in algebra to "move" negative numbers to "the other side of the equality sign" to render them positive.) His writings were also the first to take seriously that imaginary numbers ought to be contemplated as numbers.

5.5 Exercises

Exercise 5.5.1. Show that multiplication of complex numbers is an associative operation.

Exercise 5.5.2. Show that multiplication of complex numbers distributes over addition.

Exercise 5.5.3. Show that any non-zero complex number admits a multiplicative inverse.

Exercise 5.5.4. Given $z \in \mathbb{C}$, write $z = a + bi$. We define $\overline{z}$ to be the complex number $a - bi$.

Show that the function $z \mapsto \overline{z}$ (this is a function from $\mathbb{C}$ to $\mathbb{C}$) is a bijection.

Exercise 5.5.5. Compute the following:

(a) $(3 + 5i)(6 + 2i)$

(b) $(i)(3 + 2i)$

(c) i^{2022} (This is i raised to the 2022th power.)

(d) $(3 + 5i)(3 - 5i)$

(e) $(1 + 5i)(1 - 5i)$

(f) $(1 + 5i)^2$

Exercise 5.5.6. In this exercise, I will tell you the polar coordinates r' and θ' for a complex number w', and the polar coordinates r'', θ'' for a complex number w''. I want you to tell me the polar coordinates of the product $w'w''$.

(a) $r' = 1, \theta' = 0$ and $r'' = 1, \theta'' = \pi$.

(b) $r' = 2, \theta' = 0$ and $r'' = 2, \theta'' = \pi$.

(c) $r' = 2, \theta' = \pi$ and $r'' = 2, \theta'' = \pi$.

(d) $r' = 3, \theta' = \pi/2$ and $r'' = 3, \theta'' = \pi/2$.

(e) $r' = 3, \theta' = \pi/2$ and $r'' = 5, \theta'' = \pi/3$.

Exercise 5.5.7. You have seen, in a class akin to Calculus II, that the functions $e^x, \cos(x), \sin(x)$ have the following Taylor series expansions:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

(a) Even though most calculus classes only teach these series expressions when x is a real number, assume that you are allowed to plug in complex numbers for x. Under this assumption, prove that

$$e^{iy} = \cos(y) + i \sin(y)$$

for all real numbers y.

(b) (Can be tricky; have to use binomial theorem very wisely.) Assuming that you may rearrange the terms in these series, prove that

$$e^{x+iy} = e^x e^{iy}$$

for arbitrary real numbers x and y.

5.6 Extra Credit: Cross product

At this point, you have seen that $\mathbb{R}$ is a ring, and $\mathbb{R}^2$ is a ring.

You may have also seen that $\mathbb{R}^3$ admits an addition (via vector addition) and a multiplication (via cross product)—in a linear algebra class, or in a multivariable calculus class. Let me remind you that vector addition of $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$$

The cross product is given by the formula

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}.$$

You may have learned it visually as a "determinant":

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ represent the standard basis vectors. $\mathbb{R}^3$, equipped with the usual notion of addition, and with cross product as multiplication, satisfies *all but two* properties of being a ring. (I) Identify which properties fail, and prove that they fail.