Lecture 11

Exercise day

Rather than pile on more material, I would like to take this day to give you the chance to work—either on your own, or as a group—on some exercises. They will force you to look back on some definitions, and also give you the opportunity to ask questions about subtleties, or ask questions you may feel are basic, like "what am I supposed to show?"

By the way, you may want to take advantage of Proposition 10.7.3 if you ever want to show that ring homomorphism is an injection. (The proposition implies: If you can prove that a ring homomorphism f satisfies the property that $f(x) = 0 \implies x = 0$, then f is an injection.)

The problems at the end are (very fun) challenge problems.

Exercise 11.0.1.

1. Show that $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ are in bijection if and only if $m = \pm n$.
2. Show that $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ admit a ring isomorphism between them if and only if $m = \pm n$.

Exercise 11.0.2. Let $(x^2 + 1)$ be the principal ideal in $\mathbb{R}[x]$ generated by the element $x^2 + 1$.

1. Convince yourself that by sending x to the element $i \in \mathbb{C}$, you can construct a ring homomorphism

$$\mathbb{R}[x] \to \mathbb{C}$$

and that, in fact, you can construct a ring homomorphism

$$f : \mathbb{R}[x]/(x^2 + 1) \to \mathbb{C}.$$

(Hint: Theorem 10.6.1.)

2. Is f a ring isomorphism? Why or why not? It may help to remember that if a polynomial $p(x)$ has as zero at some point a, then $p(x) = q(x)(x-a)$ for some polynomial $q(x)$.

Exercise 11.0.3. (a) Show that the principal ideal (x^2-1) inside of $\mathbb{R}[x]$ is not a prime ideal. (See Exercise 10.8.1. Also, this problem may require you to factor, to some *degree*.)

(b) Is there an algebraic subset $X \subset \mathbb{R}$ for which $(x^2-1) = I(X)$? (See Notation 9.5.2.) It may help to remember that if a polynomial $p(x)$ has as zero at some point a, then $p(x) = q(x)(x-a)$ for some polynomial $q(x)$.

Exercise 11.0.4. (a) Convince yourself that the assignment $s \mapsto x$ and $t \mapsto x^2$ induces a ring homomorphism

$$\phi : \mathbb{R}[s,t] \to \mathbb{R}[x], \qquad \sum a_{i,j} s^i t^j \mapsto \sum_k \sum_{i+j=k} a_{i,j} x^{i+2j}.$$

(b) Letting I be the principal ideal $(t-s^2)$, use Theorem 10.6.1 to conclude that you have an induced ring homomorphism

$$\psi : \mathbb{R}[s,t]/(t-s^2) \to \mathbb{R}[x].$$

(c) Show that ψ is a ring isomorphism.

(d) If you believe that ring isomorphism classes detect isomorphism classes of algebraic sets, which two algebraic sets have you now shown are equivalent?

Exercise 11.0.5. (a) Convince yourself that the assignment $s \mapsto x$ and $t \mapsto p(x)$ induces a ring homomorphism

$$\phi : \mathbb{R}[s,t] \to \mathbb{R}[x], \qquad \sum a_{i,j} s^i t^j \mapsto \sum_{i,j} a_{i,j} x (p(x))^j.$$

(b) Letting I be the principal ideal $(t-p(s))$, use Theorem 10.6.1 to conclude that you have an induced ring homomorphism

$$\psi : \mathbb{R}[s,t]/(t-p(s)) \to \mathbb{R}[x].$$

(c) Show that ψ is a ring isomorphism.

(d) If you believe that ring isomorphism classes detect isomorphism classes of algebraic sets, which two algebraic sets have you now shown are equivalent?

Exercise 11.0.6. Let n be an integer. How many ring homomorphisms are there from $\mathbb{Z}/n\mathbb{Z}$ to itself?

Exercise 11.0.7. Show that there are at least two different ring isomorphisms from $\mathbb{C}$ to itself. (You'll want to *conj*ure some tricks here.)

Side notes: It turns out you can prove that these are the only two ring isomorphisms that preserve $\mathbb{R} \subset \mathbb{C}$ point-wise. There are in fact uncountably many ring isomorphisms from $\mathbb{C}$ to itself if you don't care about continuity or about preserving $\mathbb{R}$.

Exercise 11.0.8. Let $f : R \to S$ be a ring isomorphism.

(a) Show that R is an integral domain if and only if S is.

(b) Show that R is a field if and only if S is.

Side note: This shows that being a field, or an integral domain, are reasonable properties to study—they are left invariant under ring isomorphism.

Exercise 11.0.9. Let R and S be rings.

(a) Show that if R is commutative but S is not, then there does not exist any ring isomorphism between R and S.

(b) Give an example of a commutative ring R and a non-commutative ring S, together with a ring homomorphism from R to S.

Exercise 11.0.10. (a) (Challenge problem.) Does there exist a ring homomorphism from $M_2(\mathbb{R})$ to $\mathbb{R}$? How about to $\mathbb{C}$?

(b) (Challenge problem.) Does there exist a ring homomorphism from $M_3(\mathbb{R})$ to $M_2(\mathbb{R})$?