Lecture 18

Dihedral group computations and symmetric group computations

18.1 Goals

1. See the dihedral group D_{2n}—symmetries of a regular n-gon.
2. Understand that the symmetries of a regular n-gon consist of n rotations and n reflections.
3. Understand how to compute group operations in D_{2n}
4. Learn cycle notation to represent elements of S_n

18.2 So far

So far, we've seen:

(a) Examples of groups

  (1) Additive groups of rings: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, M_n(R)$.
  (2) Groups of units of rings: $\mathbb{Z}^\times, \mathbb{Q}^\times, \mathbb{R}^\times, \mathbb{C}^\times, GL_n(R)$.
  (3) Groups of geometric symmetries: Mattress group, square mattress group.

(4) Groups of "set" symmetries: Symmetric group S_n and $\mathrm{Aut}(X)$.

(5) Groups you can "find" inside of other groups (subgroups): $O_n(\mathbb{R}) \subset GL_n(\mathbb{R})$, $SL_n(\mathbb{R}) \subset GL_n(\mathbb{R})$.

(6) Groups you can build out of other groups: Product groups.

(b) Ways to find relationships between groups

(1) Group homomorphisms

(2) Group isomorphisms (which tell us ways in which two groups are "equivalent.")

(c) Properties of groups

(1) Cyclic

(2) Order of a group

(3) Order of an element

Today, we're going to learn one more important example of a group—the dihedral group—and then learn about how to represent elements of S_n. We'll begin to practice doing computations in both these groups.

18.3 Dihedral groups

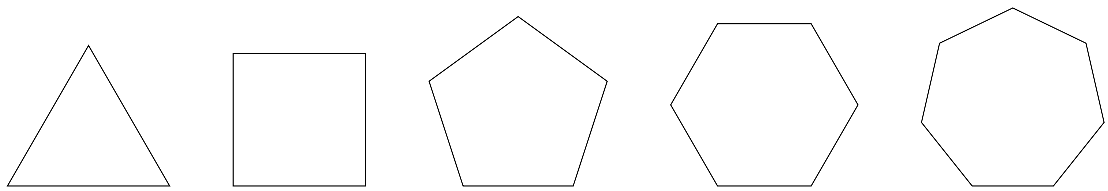
Let's continue with our tour of groups. There is a class of groups, the *dihedral groups*, which are among the simplest examples of groups.

18.3.1 Regular n-gon

Let's first define a notion we probably all know, but that I'd like to make explicit.

Definition 18.3.1. A polygon is called *regular* if all sides have the same length, and all angles have the same measure. A regular polygon with n sides (hence n vertices is called a *regular n-gon*.

Remark 18.3.2. Below are drawings of a regular 3-gon, 4-gon, 5-gon, 6-gon, and 7-gon.



A regular 3-gon is otherwise known as an isosceles triangle. A regular 4-gon is known as a square. A regular 5-gon is also known as a regular pentagon, and so forth.

Remark 18.3.3. It is important that all edges have the same length *and* all angles have the same measure.

As an example, any non-square rhombus is a parallelogram whose edges have equal length, but whose angles are not all congruent. Likewise, a non-square rectangle is a parallelogram whose angles are equal in measure, but whose edges are not of equal length. We exclude such examples—as you might be able to intuit, these shapes are not as "symmetric" as squares.

Construction 18.3.4 (Construction of a regular n-gon)**.** First, let's construct a regular n-gon—to see that a regular n-gon exists for any $n \geq 3$.

Given n, draw points on the unit circle forming angles $k(2\pi)/n$ radians with the positive x-axis. (Do this for every k between 0 and $n-1$.) As an example, if $n = 4$, you would draw points on the unit circle at angles 0, 90 degrees, 180 degrees, and 270 degrees. You have drawn a total of n points.

Now, draw the line segment between each successive pair of points. These line segments form the edges of a polygon.

Proposition 18.3.5. For all $n \geq 3$, the polygon in Construction 18.3.4 is a regular n-gon.

Proof. The construction above produces a polygon with n vertices (hence n edges). So it remains to see that the polygon is regular.

Let's observe that "rotating the plane about the origin" by any amount is an operation that preserves all distances. (If you take a segment of length l, and rotate the plane, the image of the original segment is still a segment of length l.) Moreover, rotation preserves sizes of angles. That is, if A is an angle, and if you rotate the plane by some amount, the image of A is an angle of the same measure as A.

We can apply this observation by noting that the polygon from Construction 18.3.4 has a rotational symmetry. To be concrete, let's call the polygon from the construction P. If you rotate the plane by $(2\pi)/n$ radians, the image of P is a polygon with the same set of vertices—so P! Therefore, by taking v_0 to be the vertex of P formed at 0 radians, and v_k to be the vertex of the polygon at $k(2\pi)/n$ radians, a rotation by $k(2\pi/n)$ takes v_0 to v_k; this shows that the angle measures of our polygon at v_k and v_0 are equal. Likewise, the edge from v_0 to v_1 can be taken to the edge from v_k to v_{k+1} by the same rotation. This shows that these two edges have the same length. Because k can be taken to be any integer, we see that all angles and all edges have the same length. Thus, P is regular. ☐

18.3.2 The dihedral group

Definition 18.3.6. By a *symmetry* of a regular n-gon, we mean a function from a regular n-gon to itself which preserves all distances.

Remark 18.3.7. Preserving all distances is enough to guarantee that a symmetry sends vertices to vertices. We have seen why before, but let's see it again here to save some page-flipping. Let P be a regular polygon and $f : P \to P$ a function. Let d be the distance function—given any two points p and p' on P, $d(p, p')$ is the distance between the two points. Then the distance function $d(p, p')$ is maximized when p and p' are both vertices, hence if f preserves distance—meaning $d(f(p), f(p')) = d(p, p')$ for all $p, p' \in P$—it must be that whenever p and p' are vertices, $f(p)$ and $f(p')$ are both vertices.

As a consequence, a symmetry of a polygon sends edges to edges. And because the image of an edge is determined completely by the image of its endpoints, one can understand what a symmetry of P does entirely by understanding what it does on vertices.

Remark 18.3.8. Let P and P' be two regular n-gons (for the same n). Then there is a bijection $g : P \to P'$ which preserves distance up to a single scaling factor (and preserves all angles). We then have a group isomorphism from the set of symmetries of P to the set of symmetries of P'. Indeed, given a symmetry $f : P' \to P'$, the function $g^{-1}fg$ is a symmetry of P. The assignment

$$f \mapsto g^{-1}fg$$

is the desired group isomorphism.

In other words, even if you have two non-identical regular n-gons (meaning they may consist of different points), the group of symmetries of both are equivalent (meaning they are group isomorphic).

Thus, to understand the group of symmetries of a given regular n-gon is to understand the group of symmetries of any regular n-gon. For this reason, we will often just assume our regular n-gon to be the one constructed in Construction 18.3.4.

The previous remark says that the following notation, and the use of the word "the," is justified:

Notation 18.3.9. We let D_{2n} be the group of symmetries of a regular n-gon. We call it the *dihedral group of order* $2n$.

For concreteness, we will often take D_{2n} to be the group of symmetries of the specific regular n-gon in Construction 18.3.4.

Example 18.3.10. So D_6 is the set of symmetries of an isosceles triangles. And D_8 is the group of symmetries of a square.

Here is why—even though we deal with an n-gon—the subscript is $2n$: It reminds us how many elements are in the group.

Proposition 18.3.11. There are exactly $2n$ symmetries of the regular n-gon.

Remark 18.3.12 (Any symmetry is determined by what it does on v_0 and v_1.)**.** Let P be the regular n-gon from Construction 18.3.4 and let $f : P \to P$ be a symmetry of P. By Remark 18.3.7, f is determined completely by what it does on vertices. I now claim that f is determined completely by what it does on *two* adjacent vertices. For this, let's consider the vertices v_0 and v_1. Then $f(v_0) = v_k$ for some k; and because f preserves distance, $f(v_1)$ must equal either v_{k+1} or v_{k-1}. Moreover, the image of v_0 and v_1 determines the images of all other vertices, as if $f(v_1) = v_{k+1}$, then $f(v_{k'}) = v_{k+k'}$ for all k'. Likewise, if $f(v_1) = v_{k-1}$, then $f(v_{k'}) = v_{k-k'}$. This shows the claim.

Remark 18.3.13 (Rotations and reflections)**.** Before proving Proposition 18.3.11, let us see there are at least $2n$ symmetries of a regular n-gon P. First, there are n possible symmetries given by rotation—namely, rotation by 0, $2\pi/n$, $2(2\pi)/n$, $3(2\pi)/n$, …, $(n-1)(2\pi)/n$ radians. (Rotating by 0 radians is the "do nothing" symmetry, or the identity element in the group of symmetries.)

Note that rotation by $k(2\pi)/n$ radians takes v_0 to the vertex v_k, and v_1 to the vertex v_{k+1}.

There are n other symmetries, all given by reflections. But to describe them requires understanding n-gons for even n and for odd n separately.

For any k with $0 \leq k \leq n-1$, let l_k be the line passing through v_k and bisecting the angle at v_k. When n is even, l_k passes through the vertex $v_{k+(n/2)}$. When n is odd, l_k bisects the edge opposite v_k.

Likewise, for any $0 \leq k \leq n-1$, let e_k be the perpendicular bisector to the edge between v_k and v_{k+1}.

Reflection about l_k is a symmetry of the regular n-gon. So is reflection about e_k. While it seems we thus have $2n$ reflection symmetries (reflection about $l_0, l_1, l_2, \ldots, l_{n-1}$ and $e_0, e_1, \ldots, e_{n-1}$) it turns out some of these symmetries are equal to each other.

Reflection about l_k symmetry sends v_0 to v_{2k} (where $2k$ is to be understood as an integer modulo n) and v_1 to v_{2k-1}. When n is odd, each value of k determines a distinct symmetry of the regular n-gon; so reflection about $l_0, l_1, \ldots, l_{n-1}$ produces the n distinct symmetries we desire. On the other hand, when n is odd, e_k is the same line as $l_{k+(n+1)/2}$, so no reflection about e_k is a distinct symmetry from reflection about l_k.

When n is even, we note that l_k and $l_{k+(n/2)}$ are the *same* line, so reflecting about the l_k only produces $n/2$ new symmetries. However, when n is even, no e_k is equal to any $l_{k'}$ (as the e_k do not intersect any vertices). Moreover, reflection about e_k is equal to reflection about $e_{k+(n/2)}$, so we have $n/2$ distinct reflections about the e_k lines. In sum, we have n reflections of a regular polygon with an even number of edges: $n/2$ are given by reflecting about lines bisecting angles, and $n/2$ by reflecting about perpendicular bisectors of edges.

Proof of Proposition 18.3.11. By Remark 18.3.12, any symmetry of a regular n-gon is determined completely by what the symmetry does on two adjacent vertices. In particular, there are at most $2n$ possible symmetries of P. (There are n choices for where v_0 is sent by f, and two choices thereafter of where v_1 is sent.)

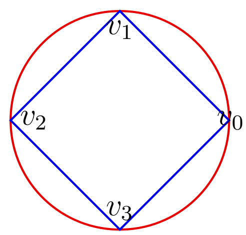
By Remark 18.3.13, there are at least $2n$ symmetries of a regular n-gon (given by n rotations and n reflections). Thus, there are exactly $2n$ symmetries. □

Remark 18.3.14. The D stands for "dihedral." Remember that polyhedra are shapes made of polygons; they typically are embedded in threedimensional shape. A "dihedron" is supposed to be a polyhedron with only two faces, which is quite a degenerate setting. In our context, whoever came up with the word "dihedral group" is imagining that our regular polygon is actually an "incredibly thin" polyhedron with only two faces (each face being the polygon). What we have referred to as a reflection of the polygon can be imagined as flipping this dihedron so that the two faces are exchanged.

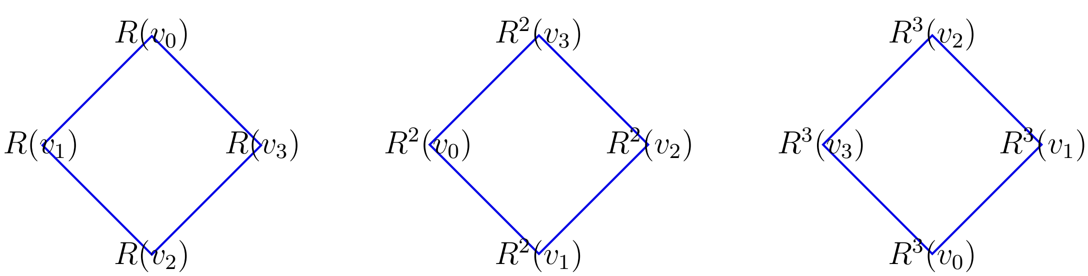
18.3.3 Pictures for D_8

For fun, let's see exactly what D_8—the group symmetries of a square—looks like.

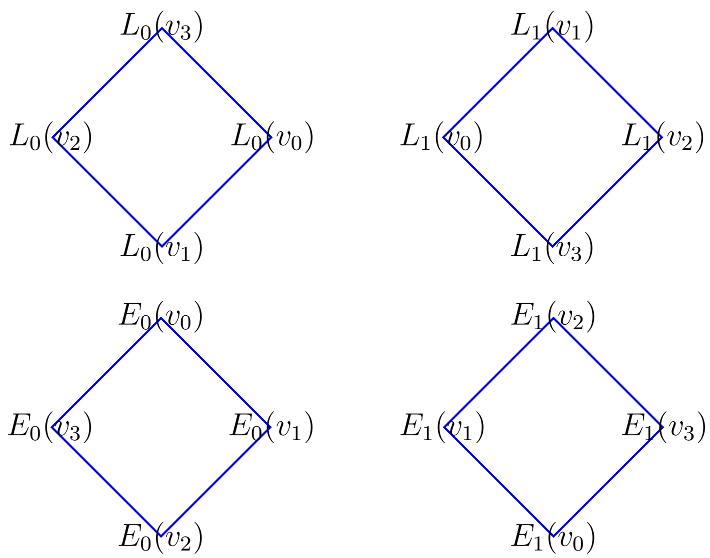
Here is a picture of the square, constructed as in Construction 18.3.4, with vertices labeled:



(The red circle is the unit circle.) Now let R denote rotation by 90 degrees counterclockwise. Here is a drawing of where the vertices $v_0, \ldots, v_3$ end up upon repeated applications of R:



Now, using the notation of Remark 18.3.13, we let L_k denote reflection about the line l_k, and E_k reflection about the line e_k. Below are pictures of where the vertices are sent under these symmetries:



Note that we have equalities

$$L_0 = L_2 \qquad L_1 = L_3 \qquad E_0 = E_2 \qquad E_1 = E_3$$

so we do not draw L_2, L_3, E_2, E_3. In other words, D_8 as a set can be written as follows:

$$D_8 = \{e, R, R^2, R^3, L_0, L_1, E_0, E_1\}.$$

There are indeed 8 elements. The first four listed are rotations; the last four are reflections.

Example 18.3.15. Let's interpret some of the drawings above. The first drawing tells us that R (rotation by 90 degrees) is a function from the square to the square having the following effects on vertices:

$$R(v_0) = v_1, \qquad R(v_1) = v_2, \qquad R(v_2) = v_3, \qquad R(v_3) = v_0.$$

The last drawing tells us that E_1 (reflection about the perpendicular bisector to the edge from v_1 to v_2) has following effects on vertices:

$$E_1(v_0) = v_3, \qquad E_1(v_1) = v_2, \qquad E_1(v_2) = v_1, \qquad E_1(v_3) = v_0.$$

Let us compose these two functions in both ways. For example, what is $E_1 \circ R$? We see from the above formulas that

$$(E_1 \circ R)(v_0) = E_1(R(v_0)) = E_1(v_1) = v_2.$$

Computing for all four vertices, and denoting $E_1 \circ R$ by the more compact E_1R, we find:

$$E_1R(v_0) = v_2, \qquad E_1R(v_1) = v_1, \qquad E_1R(v_2) = v_0, \qquad E_1R(v_3) = v_3. \tag{18.3.3.1}$$

Staring at (18.3.3.1), we have discovered that E_1R is a symmetry of the square that fixes (i.e., does not move) v_1 and v_3, but swaps v_0 and v_2. This is precisely the reflection about the line l_1 (passing through v_1 and v_3), hence the symmetry we have called L_1. So these computations finally yield the relation

$$E_1R = L_1.$$

In fact, we can also compute RE_1 (you should check the work here!) and find:

$$RE_1(v_0) = v_0, \qquad RE_1(v_1) = v_3, \qquad RE_1(v_2) = v_2, \qquad RE_1(v_3) = v_1.$$

This says RE_1 is a symmetry that fixes v_0 and v_2, but swaps v_1 and v_3. In other words, this is a reflection about the line l_0 (passing through v_0 and v_2), hence

$$RE_1 = L_0.$$

Notice that we have discovered that D_8 is *not* abelian, as $RE_1 \neq E_1R$.

18.3.4 The upshot

We have set up notation for elements of D_{2n}—an element is some power R^i of a rotation by $2\pi/n$, or some reflection L_i or E_i about the line l_i or e_i. While the notation of R, E, L is not standard outside of this course, we'll use these symbols so that we as a class have a convention for what we are talking about.

18.4 Computations in S_n

Having a very broad understanding of all the symmetries of the regular n-gon allowed us to write down elements of D_{2n} is a systematic way. Likewise, we'll now develop an understanding of the bijections of the set $\underline{n} = \{1, \dots, n\}$ to itself. This will allow us to write down elements of S_n succinctly.

18.4.1 Writing cycles

Let's set $n = 5$. It's a big enough number that bijections are pretty complicated (there are 120 of them!) but small enough that we can see what's going on.

For the sake of having an example, consider the following bijection ϕ from $\underline{n}$ to itself:

$$\phi(1) = 3, \qquad \phi(2) = 4, \qquad \phi(3) = 5, \qquad \phi(4) = 2, \qquad \phi(5) = 1. \tag{18.4.1.1}$$

As you can tell already, it's a bit annoying to process what this bijection does. It sends 1 to 3, it sends 2 to 4, et cetera, but so what? Moreover, do we really want to spend a whole *line* of text encoding a bijection? We would like a more efficient methodology and notation—just as R^n, E_k, and L_k were ways to encode a symmetry of a regular polygon.

Here's a fun way to start investigating a bijection ϕ (and, in fact, any function from a set to itself): What happens to an element when you apply ϕ to it over an over?

For example, the element 1 takes the following journey:

$$1 \overset{\phi}{\mapsto} 3 \overset{\phi}{\mapsto} 5 \overset{\phi}{\mapsto} 1.$$

Explicitly,

$$\phi(1) = 3, \qquad \phi(\phi(1)) = 5, \qquad \phi(\phi(\phi(1))) = 1.$$

Let us encode this journey in the following succinct notation:

$$(135).$$

This is called *cycle notation*, and it encodes the "cycle" (i.e., journey) that the number 1 takes under iterated applications of ϕ. The cycle (135) is notation telling us that we are considering a function that sends 1 to 3, 3 to 5, and 5 back to 1.

Using the same example of ϕ from (18.4.1.1), we can also draw the cycle of the number 2:

$$(24).$$

This notation says that 2 is sent to 4, and then 4 is sent back to 2.

18.4.2 Cycle notation for a bijection

Now, to encode the function ϕ itself, we just put the cycles together:

$$(135)(24)$$

Believe it or not, this very short, succinct sequence of symbols (135)(24) completely encapsulates the bijection ϕ.

Remark 18.4.1. Cycle notation leaves some freedoms. For example, the following are all equivalent ways to write the same cycle:

$$(135), \qquad (351), \qquad (513)$$

as they all encode some function that sends 3 to 5, 5 to 1, and 1 to 3. However, note that the cycle (153) is *not* equivalent to any of the above three. After all, (153) is a cycle for a function that sends 1 to 5; but the above three cycles depict a function that sends 1 to 3 instead.

There is *no* reason to prefer any of the above three cycles over the others. However, it is often etiquette to begin a cycle with the lowest number appearing in the cycle. So (135) is often preferred over writing (351) or (513).

Here is another freedom: The notations

$$(135)(24) \qquad \text{and} \qquad (24)(135)$$

also represent the *same* function ϕ from before. Which is to say, when writing down the cycles of a bijection, there is typically no preference given to the order in which one orders the cycles. However, as before, it is often etiquette to write down the cycles with the smallest elements appearing first, so (135)(24) would be preferred over (24)(135).

Convention 18.4.2. When $\phi(i) = i$, it is natural to write (i) for the cycle containing i. Because mathematicians are so lazy, when writing down the cycle notation for ϕ, they often leave out (i).

In other words, if some number i does not appear in the cycle notation for a bijection, this means the bijection sends i to itself.

Example 18.4.3. Consider the bijection

$$\phi(1) = 2, \qquad \phi(2) = 3, \qquad \phi(3) = 5, \qquad \phi(4) = 4, \qquad \phi(5) = 1.$$

While one could write (1235)(4) for the cycle notation representing ϕ, it is more common to simply write ϕ as

$$(1235).$$

Example 18.4.4. Let $\phi : \underline{7} \to \underline{7}$ be a bijection whose cycle notation is

$$(136)(47).$$

Then—because 2 and 5 do not appear in the cycle notation—we know that $\phi(2) = 2$ and $\phi(5) = 5$. We may further read off from the cycle notation what ϕ does on all other elements of $\underline{7}$:

$$\phi(1) = 3, \qquad \phi(3) = 6, \qquad \phi(4) = 7, \qquad \phi(6) = 1, \qquad \phi(7) = 4.$$

Example 18.4.5. No suppose we are just given the cycle notation

$$(136)(47).$$

The notation alone does not tell us what the domain (and codomain) of the corresponding bijection is. For example, it may well be that this cycle notation represents a bijection from $\underline{9}$ to $\underline{9}$, in which case the corresponding bijection fixes 8 and 9 (along with 2 and 5). Despite this ambiguity, one may appreciate that the above notation is far shorter than having to write out the clunkier

$$\text{“}(135)(2)(47)(5)(8)(9).\text{''}$$

Example 18.4.6 (The identity element)**.** So, how about the identity bijection that sends $i \mapsto i$? (This is the bijection that sends every element to itself.) Confusingly, the most common convention is to depict this bijection by the following cycle notation:

$$().$$

That's write, () is the notation for the identity function. Depending on the textbook, you may also see this being written as

$$e$$

(because it is the unit of S_n) or as

$$\text{id}$$

for “identity.”

Example 18.4.7. Let ϕ and ψ be elements of S_5 for which

$$\phi = (135)(24)$$

and

$$\psi = (314).$$

Let us compute $\phi\psi$ and $\psi\phi$. To compute $\phi\psi$, otherwise know as the composition $\phi \circ \psi$, we proceed as follows:

$$(\phi \circ \psi)(1) = \phi(\psi(1)) = \phi(4) = 2.$$

Note that to compute $\psi(1)$, we looked at the cycle notation for ψ and noted that ψ sends 1 to 4. Likewise, to compute $\phi(4)$, we looked at the cycle notation for ϕ to learn that ϕ sends 4 to 2.

This tells us we can begin the cycle notation for $\phi\psi$ by writing "(12". To write the next term, we compute

$$(\phi \circ \psi)(2) = \phi(\psi(2)) = \phi(2) = 4.$$

Above, because 2 does not appear in the cycle notation of ψ, we know that $\psi(2) = 2$. So $\phi\psi$ sends 2 to 4, and we can continue our cycle notation for $\phi\psi$ by writing "(124". We continue by computing $\phi\psi(4)$:

$$(\phi\psi)(4) = \phi(\psi(4)) = \phi(3) = 5.$$

So we have "(1245" and we now compute:

$$\phi(\psi(5)) = \phi(5) = 1.$$

This ends our cycle because 1 already appears in our cycle; so we have compute that $\phi\psi$'s cycle notation contains the cycle (1245).

Now, we may confirm that $\phi\psi(3) = 3$, to express the composite function $\phi\psi$ via the following cycle notation:

$$\phi\psi = (1245).$$

Let us compute $\psi\phi$ as well:

$$\psi(\phi(1)) = \psi(3) = 1.$$

So $\psi\phi$ fixes 1, meaning we could write (1), or just leave off (1) from the cycle notation for $\psi\phi$. Since we have computed the (trivial) cycle containing 1, let us now compute the cycle containing 2:

$$\psi(\phi(2)) = \psi(4) = 3. \qquad \psi(\phi(3)) = \psi(5) = 5. \qquad \psi(\phi(5)) = \psi(1) = 4. \qquad \psi(\phi(4)) = \psi(2) = 2.$$

So the cycle containing 2 is given by (2354). We conclude

$$\psi\phi = (2354).$$

Or, purely in cycle notation, we have:

$$(135)(24) \circ (314) = (1245)$$

and

$$(314) \circ (135)(24) = (2354).$$

18.5 Exercises

Exercise 18.5.1. The following are bijections from $\underline{7} = \{1, 2, 3, 4, 5, 6, 7\}$ to itself. Write a cycle notation representing the bijection.

(a) $\phi(1) = 3$, $\phi(2) = 1$, $\phi(3) = 6$, $\phi(4) = 5$, $\phi(5) = 4$, $\phi(6) = 7$, $\phi(7) = 2$.

(b) $\psi(1) = 3$, $\psi(2) = 1$, $\psi(3) = 6$, $\psi(4) = 4$, $\psi(5) = 7$, $\psi(6) = 2$, $\psi(7) = 5$.

(c) $\phi\psi$ (with the bijections ϕ and ψ as above.)

(d) $\psi\phi$ (with the bijections ϕ and ψ as above.)

(e) $\alpha(1) = 2$, $\alpha(2) = 1$, $\alpha(3) = 4$, $\alpha(4) = 3$, $\alpha(5) = 6$, $\alpha(6) = 7$, $\alpha(7) = 5$.

(f) $\beta(1) = 6$, $\beta(2) = 3$, $\beta(3) = 2$, $\beta(4) = 5$, $\beta(5) = 4$, $\beta(6) = 7$, $\beta(7) = 1$.

(g) $\alpha\beta$.

Exercise 18.5.2. (a) Complete the following multiplication table for the group S_3. Remember that in the row of g and the column of h should be the element gh (and *not* hg).

	e	(12)	(23)	(13)	(123)	(132)
e	e	(12)	(23)	(13)	(123)	(132)
(12)	(12)					
(23)	(23)					
(13)	(13)					
(123)	(123)					
(132)	(132)					

(b) Based on the table above, what is the order of the element (23)?

(c) Based on the table above, what is the order of the element (132)?

Exercise 18.5.3. (a) Complete the following multiplication table for the group D_6.

	e	L_3	L_1	L_2	R	R^2
e	e	L_3	L_1	L_2	R	R^2
L_3	L_3					
L_1	L_1					
L_2	L_2					
R	R					
R^2	R^2					

(b) Based on the table above, what is the order of the element R? Is there a geometric reasoning (i.e., one not using the multiplication table) for this?

(c) Based on the table above, what is the order of the element L_1? Is there a geometric way you could have inferred this?

Exercise 18.5.4. (a) Complete the following multiplication table for the group $GL_2(\mathbb{Z}/2\mathbb{Z})$. Remember that in the row of g and the column of h should be the element gh (and *not* hg).

	e	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&1\end{pmatrix}$
e	e	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&1\end{pmatrix}$
$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$					
$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$					
$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$					
$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$					
$\begin{pmatrix}0&1\\1&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&1\end{pmatrix}$					

(b) Based on the table above, what is the order of the element $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$?

(c) Based on the table above, what is the order of the element $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$?

Exercise 18.5.5. (a) Exhibit a group isomorphism from S_3 to D_6.

(b) Exhibit a group isomorphism from S_3 to $GL_2(\mathbb{Z}/2\mathbb{Z})$.

(c) Prove that if $f : G \to H$ is a group isomorphism, then so is the inverse function f^{-1}.

(d) Prove that if $f : G \to H$ is a group isomorphism and $f' : H \to K$ is a group isomoprhism, then so is the composition $f' \circ f$.

(e) Prove that D_6 is isomorphic as a group to $GL_2(\mathbb{Z}/2\mathbb{Z})$.