Lecture 20

Orbit-stabilizer theorem

20.1 Goals

1. (Terminology) Understand what an orbit of a group action is.
2. (Examples) Be able to give examples of orbits of group actions.
3. (Terminology) Understand what a stabilizer subgroup is.
4. (Conceptual) Understand what the orbit-stabilizer theorem says.
5. (Application) Apply the orbit-stabilizer theorem in examples.

20.2 Review of group actions

In the last week, we took a detour to see more examples of groups—the dihedral group and the symmetric group—and another example of group actions—conjugation.

Today, we're going to continue the study of group actions.

Let G be a group and X a set. Recall that a *left group action* of G on X is a function

$$G \times X \to X, \qquad (g, x) \mapsto gx$$

for which $ex = x$ and $g(hx) = (gh)x$.

In the following examples, we indicate the function $G \times X \to X$, but do not verify that the function satisfies the conditions of being a group action. We leave those verifications to the reader.

Example 20.2.1. Let $G = D_{2n}$ be the dihedral group of order $2n$—it is the group of symmetries of a regular n-gon P. Let X be the set of vertices of P. Then G acts on X, as any symmetry g will take any vertex v to another vertex $g(v)$, which out of sloth we may denote as gv.

Example 20.2.2. Let $G = GL_2(\mathbb{R})$. Then G acts on $X = \mathbb{R}^2$ as follows: Given any matrix $A \in G$, A sends a vector $v \in X$ to Av (defined by matrix multiplication).

Example 20.2.3. Let G be any group, and set $X = G$. Then G acts on X by conjugation. Concretely,

$$G \times X \to X, \qquad (g, h) \mapsto ghg^{-1}.$$

You should check that the pair (e, h) is sent to h, and that the pair $(g'g, h)$ has the same image as the pair (g', ghg^{-1}).

Example 20.2.4. Let G be the mattress group, and X the set of vertices (i.e., corners) of a (non-square) mattress. (See Section 14.4.) Then G acts on X—any symmetry of the mattress takes a corner of the mattress to another corner.

Here is one fact about group actions that we haven't mentioned yet, but that we'll use today:

Proposition 20.2.5. Let $G \times X \to X$ be a group action, and suppose that $gx = x'$. Then $g^{-1}x' = x$.

Proof. $g^{-1}x' = g^{-1} \cdot gx = (g^{-1}g)x = ex = x$. $\square$

In words, Proposition 20.2.5 says that g^{-1} "undoes" the action of g.

20.3 Orbits

Definition 20.3.1. Fix a group action $G \times X \to X$ and an element $x \in X$. Then the *orbit of* x is the set

$$\{gx \mid g \in G\}.$$

Equivalently, the orbit of x is the set

$$\{x' \in X \mid x' = gx \text{ for some } g \in G.\}$$

We will write the orbit of x as

$$Gx.$$

Note that Gx is a subset of X.

Remark 20.3.2. Suppose $x' \in Gx$. Then in fact, $x \in Gx'$. After all, if $gx = x'$ then $x = g^{-1}x'$. We will often say that x and x' are "in the same orbit" to reflect this symmetry—though in some instances it may be more enlightening to say that x is in the orbit of x' (or vice versa) to emphasize a particular element.

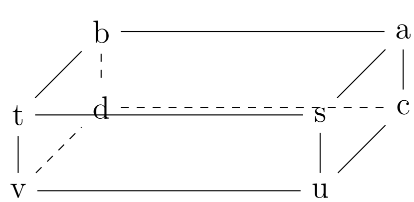
Remark 20.3.3. Given a group action $G \times X \to X$, you can think of an orbit of x as a list of all the elements of X that look "identical" from the perspective of the action. Indeed, if $gx = x'$ (so that x' is in the same orbit as x) and if you think of the action as manifesting some sort of notion of symmetry, then x and x' look identical from whatever symmetry the action is trying to express.

Remark 20.3.4. Given a group action $G \times X \to X$, one can think of X as "split up" into its orbits. In other words, X is a disjoint union of its orbits. From the perspective of the previous remark, we are splitting X up into groups of elements that look similar to each other (with respect to the action).

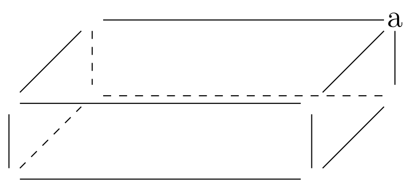
Warning 20.3.5. Given a group G and a set X, there may be many different actions of G on X. So you should make sure that, in each of the previous remark, you understand that the symmetry is not something inherent to the sets G and X, but rather only manifested once one specifies an action of G on X.

Example 20.3.6 (The mattress group's action on vertices)**.** Let's dig into the example of the mattress group. Remember that the mattress group consists of four elements: e (do nothing), R_x (rotate 180 degrees about the x-axis), R_y (rotate 180 degrees about the y-axis), and R_z (rotate 180 degrees about the z-axis).

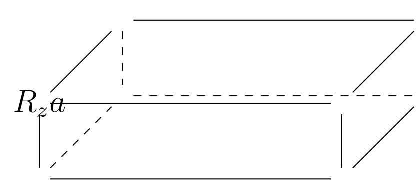
Let's (very creatively) label the elements of the vertex set a, b, c, d, s, t, u, v as follows:



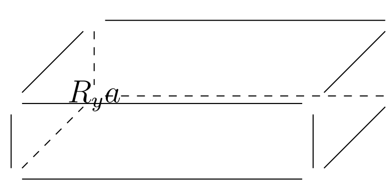
Let's understand the orbit of the element a. Here are the images of a under the four elements of G:



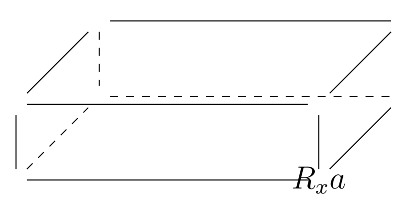
e (Do nothing)



R_z (Rotate 180° about z-axis)



R_y (Rotate 180° about y axis)



R_x (Rotate 180° about x axis)

In other words, we see that $ea = a$, $R_z a = t$, $R_y a = d$, $R_x a = u$. So the orbit of a can be written

$$Ga = \{a, d, t, u\}.$$

So the orbit of a is a set of size 4, not the entire set X. So we see here an example where an orbit of an element need not be the entire X.

In fact, the set X can be split up into exactly two orbits: Ga and Gb.

Remark 20.3.7. You can in fact check that the elements $a, d, t, u \in X$ have the same orbits:

$$Ga = Gd = Gt = Gu.$$

The pattern to observe: Two elements of X are in the same orbit if and only if their orbits are equal. You will verify this in Exercise 20.7.1 below.

Example 20.3.8. Let P be a regular n-gon and X its set of vertices. Letting $G = D_{2n}$ be the dihedral group, we have an action $G \times X \to X$. How many orbits does this action have?

Any vertex v of the polygon may be taken to any other by a rotation, so in fact, this action has only one orbit: X itself.

Example 20.3.9. Let $G = GL_2(\mathbb{R})$ and $X = \mathbb{R}^2$ with the usual action by matrix multiplication. You can prove that any non-zero vector $v \in \mathbb{R}^2$ can be taken to any other non-zero vector v' by a well-chosen invertible matrix A. On the other hand, if A is invertible, $Av = 0$ if and only if $v = 0$. So this action consists of exactly two orbits: The orbit

$$G0 = \{0\}$$

consisting of only one element (the origin), and another orbit

$$\mathbb{R}^2 \setminus \{0\}$$

consisting of all non-zero vectors.

20.4 Stabilizers

Given a group action on X and an element x of X, we have defined a subset of X called the orbit of x. We will now define a subgroup of G associated to x.

Definition 20.4.1. Let $G \times X \to X$ be a group action and fix $x \in X$. The *stabilizer* of x is the collection of elements of G that act trivially on x. Concretely, the stabilizer of x is the set

$$\{g \in G \,|\, gx = x\}.$$

We denote the stabilizer of x by

$$G_x.$$

Proposition 20.4.2. Let $G \times X \to X$ be a group action and fix $x \in X$. Then G_x is a subgroup of G.

Proof. We must check three conditions: G_x contains e, (ii) If $g \in G_x$ then $g^{-1} \in G_x$ (so G_x is closed under inverses) and (iii) If $g, g' \in G_x$ then $gg' \in G_x$ (so G_x is closed under products).

(i) By definition of group action, $ex = x$. Thus $e \in G_x$.

(ii) By definition of group action,

$$g^{-1}(gx) = (g^{-1}g)x = ex = x.$$

On the other hand, if $gx \in G_x$, we see that $gx = x$, hence we conclude

$$g^{-1}x = x.$$

This shows $g^{-1} \in G_x$.

(iii) The definition of group action tells us

$$(gg')x = g(g'x).$$

Assuming $g' \in G_x$, we see $g'x = x$, so

$$(gg')x = g(g'x) = gx.$$

Assuming further that $g \in G_x$, we see $gx = x$, so

$$(gg')x = x.$$

By definition, then, $gg' \in G_x$. This completes the proof. □

Remark 20.4.3. So, finding group actions of G on a set X can also lead to finding interesting subgroups of G—given any $x \in X$, the stabilizer G_x has a chance at being a (fun) subgroup of G.

Example 20.4.4. Let G be the mattress group and X the set of corners of the mattress. Fix a corner, say a (using the notation from Example 20.3.6). What is the stabilizer?

Obviously, G_a contains e, as "do nothing" fixes a. But all other symmetries of the mattress move a. Thus, the stabilizer of a is actually a trivial subgroup, consisting of only the identity element:

$$G_a = \{e\}.$$

Example 20.4.5. What about the action of D_{2n} on the set X of vertices of the regular n-gon? Well, let's fix a vertex $v \in X$. There are only two symmetries of the regular n-gon that fix v: e (do nothing), and reflection L_v about the angle bisector at v. So the stabilizer of v is a subgroup consisting of exactly two elements:

$$G_v = \{e, L_v\}.$$

So far, regardless of $x \in X$, the stabilizers G_x have been isomorphic (though they are not the *same* subsets of G). Here is an action where stabilizer subgroups look drastically different:

Example 20.4.6. Let $G = GL_2(\mathbb{R})$ and $X = \mathbb{R}^2$. The zero vector $0 \in \mathbb{R}^2$ of course fixed by any matrix, so the stabilizer of 0 is all of G:

$$G_0 = G.$$

On the other hand, let $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be the standard basis vector with first coordinate 1. Then the stabilizer of v is the collection of all invertible matrices whose first column is v:

$$G_v = \{ \begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix} \mid c \neq 0.\}$$

20.5 The orbit-stabilizer theorem

Given a group action $G \times X \to X$ and an element $x \in X$, we can associate:

(a) A subset Gx of X called the orbit of x.

(b) A subgroup G_x of G called the stabilizer of x.

(Beware of whether lower-case x is a subscript, or is on the same line as G.)

As its name implies, the orbit-stabilizer theorem relates the two notions—specifically, the sizes of these two notions.

Notation 20.5.1. For any finite set S, let $|S|$ denote the number of elements in S. (When G is a finite group, $|G|$ thus denotes the order of G.)

Theorem 20.5.2. Suppose G is a finite group. Fix any group action $G \times X \to X$, and any element $x \in X$. Then

$$|G| = |Gx| \times |G_x|.$$

In other words, the number of elements in G is a product of the number of elements in the orbit of x and the number of elements in the stabilizer of x.

Proof. We will write G as a disjoint union of $|Gx|$ many subsets, each of size $|G_x|$; this will prove the claim.

Because G is finite, so is Gx. So let's label the elements of Gx:

$$Gx = \{x_1, x_2, \ldots, x_k\}$$

where $k = |Gx|$ is the number of elements in G_x. For the sake of concreteness, let's assume $x_1 = x$. And for each x_i, let

$$C_i := \{g \in G \mid gx = x_i\}.$$

That is, C_i is the set of all elements in G taking x to x_i. For example, $C_1 = G_x$ is the stabilizer of x (because we've declared $x_1 = x$).

The $C_1, \ldots, C_k$ form a partition of G. Let's note that every element of G is in C_i for some i—because an element of g takes x somewhere, and by definition, takes x somewhere in the orbit of x.

Moreover, if $i \neq j$, then $C_i \cap C_j = \emptyset$, because no group element can take x to two *different* elements. So indeed, the sets $C_1, \ldots, C_k$ partition G, meaning that

$$|C_1| + |C_2| + \ldots + |C_k| = |G|. \tag{20.5.0.1}$$

Each C_i admits a bijection to C_1. I now claim that, for every i, there is a bijection from C_1 to C_i. To see this, let's choose an element $g \in G$ for which $gx = x_i$. (Such a g exists by the assumption that x_i is in the orbit of x.) Fixing this g once and for all, I claim that "multiplication by g on the left" defines a function

$$l_g : C_1 \to C_i, \qquad h \mapsto gh.$$

Let's check that r_g has codomain C_i: If $h \in C_1$, we know $hx = x$, so $(gh)x = g(hx) = gx = x_i$.

Now I claim l_g admits an inverse function, namely $l_{g^{-1}}$. To see this, let's first note that $g^{-1}(x_i) = x$ (Proposition 20.2.5); this shows that $l_{g^{-1}}$ takes C_i to C_1. To see that $l_{g^{-1}}$ is an inverse to l_g, let's verify it's a two-sided inverse:

$$(l_{g^{-1}} \circ l_g)(h) = l_{g^{-1}}(gh) = g^{-1}(gh) = eh. \qquad (l_g \circ l_{g^{-1}})(h) = gg^{-1}h = h.$$

This last bolded statement tells us that $|C_i| = |C_1|$ for every i, so the equality (20.5.0.1) becomes

$$|C_1| + \ldots + |C_1| = |G|$$

where the summation has k terms in it. In other words,

$$k|C_1| = |G|.$$

Now, remembering that $k = |Gx|$ is the size of the orbit, and $C_1 = G_x$ is the stabilizer of x, the result follows. $\square$

20.6 Some minor applications of the orbit-stabilizer theorem

Big theorems are always good for checking answers.

Example 20.6.1. The dihedral group D_{2n} acts on the set of vertices of a regular n-gon. Pick a vertex v. Its orbit consists of n elements (because an n-gon has n vertices, and any vertex can be taken to any other by a rotation). Moreover, the stabilizer of v has order 2 (because the only symmetries that fix a given vertex are e, and reflection about the angle bisector at v). So the orbit-stabilizer theorem tells us that the order of G is given by

$$|Gv| \times |G_v| = n \times 2 = 2n.$$

This verifies that $2n$ is indeed the order of the group of symmetries of the regular n-gon.

Example 20.6.2. Let $G = S_n$ be the symmetric group on n letters. As you know, S_n acts on $\underline{n} = \{1, 2, \ldots, n\}$. Given any two elements $i, j \in \underline{n}$, you can find a bijection that sends i to j (for example, just swap i and j and leave all other elements fixed). So this group action consists of one orbit, of size n.

So pick an eleement $i \in \underline{n}$. How big is its stabilizer? (In other words, how many bijections are there that send I to itself?) The orbit stabilizer-theorem tells us

$$|G| = |Gi| \times |G_i|$$

where $G = S_n$, G_i is the stabilizer of i, and Gi is the orbit. Well, we have

$$n! = n \times |G_i|$$

So we conclude that the stabilizer has size $(n-1)!$.

This isn't so surprising. In fact, taking $i = n$, the stabilizer G_n is just the collection of permutations that leave n fixed; in other words, the number of ways you can permute $\{1, \ldots, n-1\}$. And of course there are $(n-1)!$ ways of doing this.

But! You can also use the orbit-stabilizer theorem to have deduced that S_n has size n factorial, by induction.

First, note that when $n = 1$, obviously S_1 has size 1.

By induction, suppose that you have proven that S_{n-1} has size $(n-1)!$. Then the stabilizer of n can be identified wish S_{n-1}, so we conclude

$$|S_n| = n \times |S_{N-1}|$$

by the orbit-stabilizer theorem. By induction, we conclude

$$|S_n| = n \times (N-1)! = n!.$$

20.7 Exercises

Exercise 20.7.1. Let $G \times X \to X$ be a group action.

(a) Show that the relation "is in the orbit of" is an equivalence relation. More precisely, declare that $x \sim x'$ if and only if x' is in the orbit of x. Show that $\sim$ is an equivalence relation.

(b) Show that an orbit of the group action is precisely an equivalence class of the above relation.

Then, the next two parts of this problem are completely formal (and require no knowledge of groups):

(c) Show that if x' is in the orbit of x, then $Gx = Gx'$.

(d) Show that if $Gx = Gx'$, then x' is in the orbit of x.

Exercise 20.7.2. Verify the stabilizer computations in Example 20.4.6. That is, compute the stabilizer of $0 \in \mathbb{R}^2$, and of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, for the action of $G = GL_2(\mathbb{R})$ on $X = \mathbb{R}^2$.

Example 20.7.3. Suppose that G is a group of prime order. (This means $|G|$ is a prime number.) Show that if G acts on a set X, the orbits of the group action can only have sizes 1 or $|G|$.

Example 20.7.4. Let P be a *polyhedron* (not a polygon). And suppose that P has the following symmetry properties:

(i) Given any two faces of P, there exists a symmetry of P that takes one face to the other. (In particular, every face of P is congruent to any other.)

(ii) Every face of P is a regular n-gon.

(iii) Moreover, rotation by $2\pi/n$ about an axis perpendicular to a face is a symmetry of P.

Let G be the group of rotational symmetries of P. Let X be the set of *faces* of P.

(a) Show that G has order $n \times |X|$.

(b) Compute the number of rotational symmetries of a cube.