Lecture 24

Exploration day: Linear fractional transformations

24.1 Goals

1. Explore an interesting geometric space—upper half plane.
2. Explore how linear fractional transformations are symmetries of the upper half plane.

24.2 Upper half plane

Definition 24.2.1. The *upper half plane* $\mathbb{H}$ is the set of all complex numbers z whose imaginary part is positive. In other words,

$$\mathbb{H} := \{x + iy \in \mathbb{C} \mid y > 0\}.$$

With no motivation whatsoever (see Remark 24.2.3), let me introduce the following terminology:

Definition 24.2.2 (The two kinds of geodesics in the upper half plane)**.** We say that a curve $C \subset \mathbb{H}$ is a *geodesic* if

(a) C is a vertical half-line. That is, there is some real number x_0 for which

$$C = \{x_0 + iy \mid y > 0\}.$$

or

(b) C is a semicircle with diameter on the real line. Put another way, C is an arc of a circle, and C forms right angles with the x-axis.

Remark 24.2.3. It turns out that one can endow upper half-space with a *hyperbolic metric*, whatever that is. In this geometry, it turns out a shortest path between two points is not always a line; in general, the shortest path between two points is a curve of one of the types laid out in Definition 24.2.2.

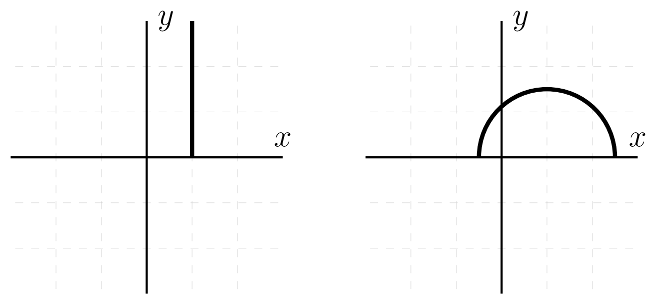

Figure 24.1: Examples of geodesics in upper half-plane

24.3 Linear factional transformations

Construction 24.3.1. Choose four real numbers a, b, c, d and assume that $ad - bc > 0$. This defines the following transformation: Given a complex number z, send z to the complex number

$$\frac{az+b}{cz+d}. \tag{24.3.0.1}$$

Remark 24.3.2. Even when a, b, c, d are complex numbers, any function taking z to (24.3.0.1) is called a *linear fractional transformation*. Such transformations are also called *Möbius transformations*.[1] Today, we will only talk about linear fractional transformations where a, b, c, d are real numbers and $ad - bc > 0$.

[1] August Ferdinand Möbius was a German mathematician active in the first half of the 1800's. He is the same person after whom the famous strip/band is named. You may have also heard of his inversion formula in number theory/combinatorics. As far as I understand, he discovered one fundamental idea after another in both astronomy and in mathematics; he was a student of the even more famous Gauss.

Warning 24.3.3. A linear fractional transformation is *not* a linear transformation. In fact, though tradition dictates the term "linear fractional transformation," you should imagine a hyphen: "linear-fractional transformation." The word linear is not modifying the word transformation; rather, it is telling you what kind of fraction represents the transformation.

Exercise 24.3.4 (Visualization. Mandatory)**.** I wasn't able to find the best visualizations online for our purposes. Here is one tool[2] you can use to play:

`https://www.desmos.com/calculator/snbnnk6wkw`

I warn you that the numbers a, b, c, d are not constrained to have $ad - bc > 0$ in this link. However, by default, the demo should display the transformation

$$f : z \mapsto \frac{z+2}{z+4}$$

(which is the case $a = 1, b = 2, c = 1, d = 4$).

(a) Drag the red dot along a vertical line. Convince yourself that vertical lines are sent to circles whose diameters lie along the real axis. (In other words, in this example of f, vertical geodesics are sent to circular geodesics.)

(b) Drag the red dot along a horizontal line. Convince yourself that all horizontal lines are sent to circles tangent to *the same point* along the x-axis. (This is just a fun thing to observe; the technical term is that *horocycles* are preserved, but we won't discuss horocycles any further.)

(c) Play around changing a, b, c, d parameters. (Remark: $c = 0$ is the "most boring" case.)

Exercise 24.3.5 (Mandatory)**.** Let $a = d = 1$, $c = 0$, and b any real number. Show that the function

$$z \mapsto \frac{az+b}{cz+d}$$

is a function that you can think of translation—more precisely, as a function "shifting elements by b units in the real direction."

[2] Attribution: I made this Desmos tool based on a tool created by a reddit user. The source is here: `https://www.reddit.com/r/desmos/comments/m03w1t/complex_function_visualizer/`

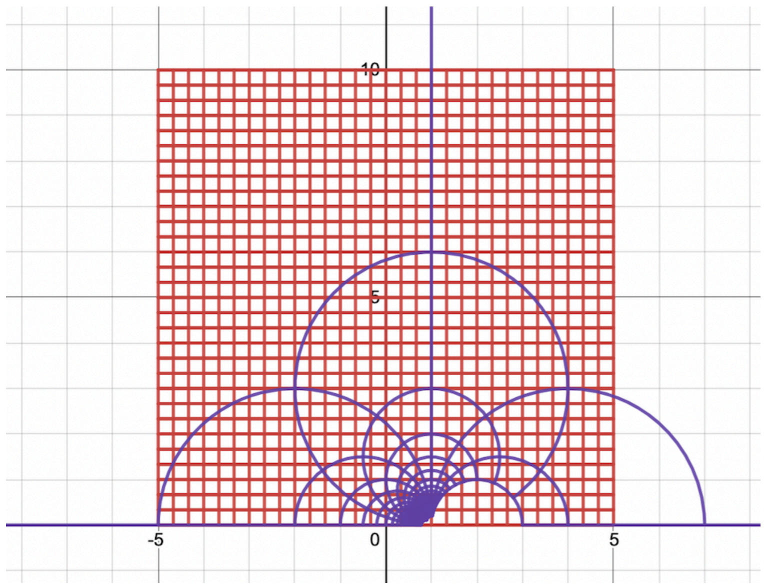

Figure 24.2: The linear fractional transformation $z \mapsto \frac{z+2}{z+4}$, taking the red grid to the blue "grid".

Exercise 24.3.6 (Mandatory)**.** Let $b = c = 0$, $d = 1$, and a any real number. Show that the function

$$z \mapsto \frac{az+b}{cz+d}$$

is a function that you can think of scaling—more precisely, as a function "dilating elements by a units radially."

Exercise 24.3.7 (Optional)**.** Let $a = d = 0$, $b = 1$, and c any real number. Show that the function

$$z \mapsto \frac{az+b}{cz+d}$$

is a function that you can think of inversion about a circle of radius c.

It turns out that any linear fractional transformation (with $ad - bc > 0$ and $a, b, c, d \in \mathbb{R}$) is a bijection from $\mathbb{H}$ to itself (Exercise 24.4.5). Moreover, this bijection is not any old bijection, but is a bijection of $\mathbb{H}$ that preserves geodesics. In other words, it is a symmetry of $\mathbb{H}$ *as a space with some notion of geodesics*, and not just as a set.

Exercise 24.3.8 (Optional)**.** Suppose that C is a geodesic, and $f : \mathbb{H} \to \mathbb{H}$ is a linear fractional transformation with $ad - bc > 0$ and $a, b, c, d \in \mathbb{R}$.

(a) Show that $f(C)$ is also a geodesic.

(b) Moreover, show that f preserves the angles of intersection between any two geodesics.

Let me remark that, at this stage, it would be very difficult to prove the above statements purely geometric (because we haven't developed, in this course, a useful geometric interpretation of linear fractional transformations). So algebra is your only tool. Thankfully, we know how to write down algebraic equations representing angles (e.g., using dot product) and circles.

In fact, we have the following:

Exercise 24.3.9 (Very tedious, you should skip it)**.** Suppose that $f : \mathbb{H} \to \mathbb{H}$ is a bijection that sends geodesics to geodesics. Prove that f must be a linear fractional transformation with $ad - bc > 0$ and $a, d, b, c \in \mathbb{R}$.

This exercise says that, if you consider a symmetry of $\mathbb{H}$ to be a bijection preserving geodesics, then every symmetry is a linear fractional transformation. Motivated by this, we set the following notation:

Notation 24.3.10. We let $\mathrm{Aut}(\mathbb{H})$ denote the set of linear fractional transformations with $ad - bc > 0$ and $a, d, b, c \in \mathbb{R}$.

Warning 24.3.11. Two different choices of a, b, c, d can give rise to the *same* linear fractional transformation. For example,

$$f(z) = \frac{2z+3}{-z+4} \qquad \text{and} \qquad g(z) = \frac{4z+6}{-2z+8}$$

are the *same* function. (Make sure you understand why.)

24.4 Isomorphism with $PSL_2(\mathbb{R})$

It turns out that linear fractional transformations form a group. Let's take things one thing at a time.

Exercise 24.4.1 (Verifying that composition of linear fractional transformations is again linear fractional)**.** Let $f : \mathbb{H} \to \mathbb{H}$ and $g : \mathbb{H} \to \mathbb{H}$ be linear fractional transformations. Show that $g \circ f$ is a linear fractional transformation.

Notation 24.4.2. Let $SL_2(\mathbb{R})$ denote the group of those 2-by-2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfying $ad - bc = 1$.

Exercise 24.4.3 (Mandatory)**.** Define a function $\phi : SL_2(\mathbb{R}) \to \mathrm{Aut}(\mathbb{H})$ (Notation 24.3.10) by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az+b}{cz+d} \right).$$

Show that for two elements $A, B \in SL_2(\mathbb{R})$, we have

$$\phi(AB) = \phi(A) \circ \phi(B).$$

In other words, ϕ respects the binary operations of $SL_2(\mathbb{R})$ and of $\mathrm{Aut}(\mathbb{H})$.

Exercise 24.4.4 (Mandatory)**.** Let ϕ be as in the previous exercise.

(a) Show that ϕ is a surjection.

(b) Show that ϕ of the identity matrix is the identity linear fractional transformation ($a = d = 1$ and $b = c = 0$).

(c) Explain why the above—along with Exercise 24.4.3—is enough to tell you that $\mathrm{Aut}(\mathbb{H})$ is a group under composition of transformations.

Exercise 24.4.5 (Mandatory)**.** Choose four real numbers a, b, c, d and assume that $ad - bc > 0$. Show that the function

$$z \mapsto \frac{az+b}{cz+d}$$

is a bijection from $\mathbb{H}$ to itself. (Hint: Does the previous exercise allow you to construct an inverse function to this linear fractional transformation?)

Exercise 24.4.6. Let $K = \{I, -I\} \subset SL_2(\mathbb{R})$. You may take for granted that K is a normal subgroup. Consider the quotient group

$$PSL_2(\mathbb{R}) := SL_2(\mathbb{R})/K.$$

Explain, using the first isomorphism theorem (Theorem 23.6.3), why $PSL_2(\mathbb{R})$ is isomorphic to $\mathrm{Aut}(\mathbb{H})$. (You will want to identify the kernel of ϕ.)

Remark 24.4.7. The above exercises led you to one of the more useful facts in math. A very special and mysterious space—the upper half plane with today's notion of geodesics (also known as hyperbolic space) has a group of symmetries that admits an incredibly concrete description: As a very straightforward quotient of some group of matrices.

Remember, we should think of matrices as computable objects. If I give you two matrices, I'm willing to bet a hundred dollars that you can multiply them together. At the same time, it also turns out that groups like $SL_2(\mathbb{R})$ and $PSL_2(\mathbb{R})$ are hard to study explicitly. For example, if somebody asked you to classify its subgroups, it's a daunting task. It turns out that the isomorphism of $PSL_2(\mathbb{R})$ with $\mathrm{Aut}(\mathbb{H})$ allows us to study subgroups of $PSL_2(\mathbb{R})$ based on the *geometry* of how a subgroup acts on $\mathrm{Aut}(\mathbb{H})$. This interplay, probably first exploited by Poincaré when studying complex analysis, is a fundamental building block in many areas of modern math.

24.5 Some geometric properties

24.5.1 Transitivity

First, let's see that $\mathbb{H}$ is very symmetric, in that any point can be taken to another point using an element of $\mathrm{Aut}(\mathbb{H})$:

Exercise 24.5.1. Let $z, z' \in \mathbb{H}$. Show that there exists an element $f \in \mathrm{Aut}(\mathbb{H})$ for which $f(z) = z'$.

A group action $G \times X \to X$ taking any $x \in X$ to any $x' \in X$ (i.e., for any $x, x' \in X$, there exists $g \in G$ for which $gx = x'$) is called *transitive*. It shows that, whatever symmetries of X is manifested by the group action, the symmetries are so symmetric that any two elements of X can be transported, one to the other, by a symmetry g.

24.5.2 Stabilizer of i

Exercise 24.5.2 (A fun one!)**.** Let $G \subset \mathrm{Aut}(\mathbb{H})$ be the stabilizer of i (the square root of -1 contained in the upper half plane). In other words, G is the set of all functions $f(x) = \frac{az+b}{cz+d}$ (with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$) for which $f(i) = i$.

Explain why you can identify G with a circle (or, confusingly, with a circle whose antipodal points are identified).