Lecture 26

Symmetries of a cube

26.1 Goals

1. Get practice using group actions to figure out orders of groups
2. See an example of relating a geometrically defined group (symmetries of a cube) to a group where we can perform computations more quickly

26.2 Counting symmetries

Let's count the number of rotational symmetries that the platonic solids have. For this, it will help if you've played Dungeons and Dragons, or otherwise used dice with 4, 6, 8, 12, or 20 faces.

Definition 26.2.1. Let P be a polyhedron (which we think of as a subset of $\mathbb{R}^3$). A *rotational symmetry* of P is rotation of $\mathbb{R}^3$ that sends P to P. Put another way, a rotational symmetry of P is a bijection from P to itself that can be expressed as a rotation in three-dimensional space.

Example 26.2.2 (Number of rotational symmetries of a cube)**.** Imagine you have a cube—or a 6-sided die—sitting on a table. How many distinct ways can you re-orient the cube? Put another way, if you imagine the cube is a (very strangely designed) mattress, how many ways are there to reset the mattress into its frame?

Here is one way to count the number of orientations of a cube: There are exactly 6 different faces that can face upward, and one you know which face

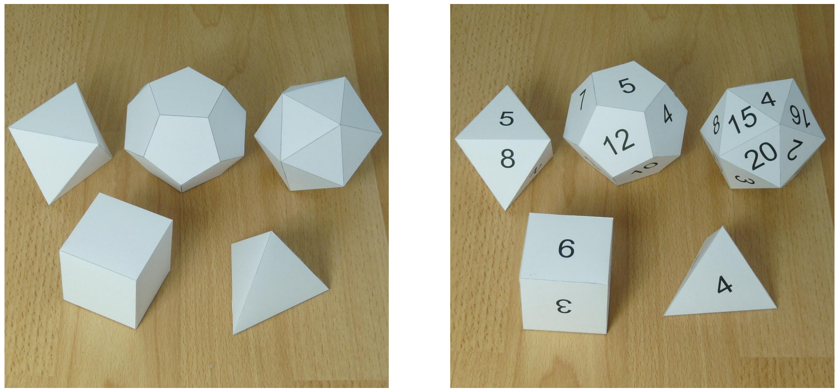


Figure 26.1: The five platonic solids, taken from `www.polyhedra.net`. The righthand image labels some of the faces to help distinguish various symmetries.

of a cube is facing upward, you only need to know which of the four adjacent faces is facing a fixed direction. Six times four is twenty-four, so there are twenty-four possible orientations of the cube. You can find a list of all 24 orientations at this image, where the cube is labeled as a standard six-sided die for ease of visualization:

`https://www.istockphoto.com/vector/isometric-icons-of-3d-dice-set-gm957782852-26153`

More generally, suppose you have a regular polyhedron/die P with F faces, each of which is a regular k-gon. When you roll the die, there are exactly F faces that could end up face down on the table. Once you know that your die is oriented so that a particular face is touching the table, there are exactly k rotations you can perform (on the face touching the table, and hence on the polyhedron itself).

Remark 26.2.3. If your polyhedron were not regular, this rotation might not send P back to itself; but because P is regular, this rotation does send the set P to the set P.

Let us assume that the following regular polyhedra exist. (They do, but proving that they exist is quite involved.) They are also called the platonic solids.

- The tetrahedron, which has 4 faces, each of which is an equilateral triangle.
- The cube, which has 6 faces, each of which is a square.
- The octahedron, which has 8 faces, each of which is an equilateral triangle.
- The dodecahedron, which has 12 faces, each of which is a regular pentagon.
- The icosahedron, which has 20 faces, each of which is an equilateral triangle.

Then the preceding discussion shows the following:

Proposition 26.2.4. The platonic solids have the following number of rotational symmmetries:

- The tetrahedron has $4 \times 3 = 12$ rotational symmetries.
- The cube has $6 \times 4 = 24$ rotational symmetries.
- The octahedron has $8 \times 3 = 24$ rotational symmetries.
- The dodecahedron has $12 \times 5 = 60$ rotational symmetries.
- The icosahedron, has $20 \times 3 = 60$ rotational symmetries.

26.3 Identifying the group of symmetries of the cube

Proposition 26.2.4 tells us that the group of rotational symmetries of a cube is some group of order 24. But is it a group we can understand?

Here is one strategy for trying to understand a group G. (i) Find a set X on which G acts. (ii) Write down the group homomorphism $G \to \mathrm{Aut}(X)$ encoding the group action. (iii) Study the image and kernel of this homomorphism.

Example 26.3.1. For example, if the homomorphism $G \to \mathrm{Aut}(X)$ has a non-trivial kernel, you have found a non-trivial normal subgroup of G, which is already a huge accomplishment. On the other hand, if the kernel is trivial, then you have identified G with a subgroup of $\mathrm{Aut}(X)$.

Warning 26.3.2. In general, it is very hard to study subgroups of $\mathrm{Aut}(X)$, but you at least have a concrete understanding of G. (When X is a finite set with n elements, you can perform concrete computations in the symmetric group $\mathrm{Aut}(X) \cong S_n$.)

In fact, any finite group is a subgroup of $S_{|G|}$. This is called Cayley's theorem, and can be proven by studying the action of G on itself by multiplication.

Discussion time. I'd like you to spend the next thirty minutes or so exploring various sets that the group of rotational symmetries of a cube acts on.

Let G be the group of rotational symmetries of a cube. We already know $|G| = 24$. Incidentally, 24 is a factorial; so we might hope that G is isomorphic to a symmetric group (specifically, that $G \cong S_4$).

So, is there some geometrically defined collection of four objects on which G acts?

It takes some thinking, but here is one: The set of diagonals of the cube. More precisely:

Definition 26.3.3 (Diagonals)**.** Given any vertex v of the cube, let $\overline{v}$ denote the vertex of the cube farthest away from v. The segment between these two vertices is called the *diagonal* between v and $\overline{v}$.

Remark 26.3.4. How many diagonals are there? There is one for every pair of opposite vertices; hence there are half as many diagonals as there are vertices. Since the cube has 8 vertices, we conclude there are 4 diagonals.

Remark 26.3.5. Let Δ be the set of diagonals of the cube. (There are 4 elements in Δ.) Let's make sure that G acts on Δ. Indeed, if $g \in G$ is a rotational symmetry of the cube, we are guaranteed that g sends vertices to vertices. On the other hand, g must preserve distances—meaning $dist(x, x') = dist(gx, gx')$ for every pair of points x, x' in the cube—so if $\overline{v}$ is the point farthest from v, it follows that $g\overline{v}$ is the point farthest from gv. In particular, g of a diagonal will again be a diagonal.

So we have a function $G \to \text{Aut}(\Delta)$ given by the group action of Remark 26.3.5. By choosing a labeling of the diagonals, let's identify Δ with the set $\{1, 2, 3, 4\}$ to we can identify $\text{Aut}(\Delta)$ with S_4. We thus have a group homomorphism

$$G \to S_4. \tag{26.3.0.1}$$

Proposition 26.3.6. The above group homomorphism is an injection.

Proof. It suffices to check that the kernel is trivial. So suppose that g is a symmetry of the cube for which every diagonal is sent to itself. This then implies (make sure you think about why) that g must send every vertex v to itself, or send every vertex v to its opposite $\overline{v}$. But there is no rotation of Euclidean space that takes the vertices of the cube and send each to their opposite. (See Lemma 26.3.7 below.)

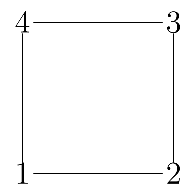
So we have seen that if g is in the kernel, $g = e$. This completes the proof. $\square$

We needed a geometric fact (isn't that satisfying?) in proving the algebraic proposition above. Let's see the Lema:

Lemma 26.3.7. There is no rotation of the cube which sends every vertex v to its opposite $\overline{v}$.

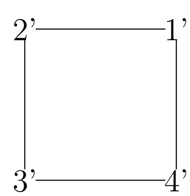
Proof. Choose your favorite face f of the cube. For sake of concreteness, let's imagine that the cube is sitting flatly on a table, and that f is the face opposite the table—i.e, the face you can see by looking down at the cube.

Now, as you look down at the cube (and hence see the face f), let's label your favorite vertex of this face as 1, and then keep labeling in counter-clockwise order 1, 2, 3, 4:



Now note that, regardless of what rotational symmetry g we apply to the cube, if we consider the face $g(f)$ and look at the face from outside the cube, the vertices of $g(f)$ can be labeled $g(1), g(2), g(3), g(4)$ in counter-clockwise order.

However, suppose g is a symmetry that sends every vertex v to its diagonal opposite $\overline{v}$. (Necessarily, the face f is then sent by g to the face opposite f—the unique face not adjacent to f.) Then, looking at $g(f)$ from outside the cube, the vertices of $g(f)$ must be labeled as follows:



This is not in counterclockwise order; hence there is no rotation g acting this way on vertices. □

Let's recoup. We're studying a group homomorphism

$$G \to \mathrm{Aut}(\Delta) \cong S_4$$

where Δ is a set with four elements: the four diagonals of the cube. By enumerating the four diagonals, we identify the group of bijections $\text{Aut}(Delta)$ with S_4.

Proposition 26.3.6 tells us that the above homomorphism is an injection. On the other hand, both the domain and the codomain have order 24.[1] Any injection between finite sets with the same cardinality must be a bijection, so in particular, we conclude:

Theorem 26.3.8. The group of rotational symmetries of a cube is isomorphic to S_4.

26.4 Using the identification

Theorem 26.3.8 is very useful.

Example 26.4.1. For one thing, I think you're kind of tired of writing out group multiplication tables. It would have been rather annoying, and time-consuming, to write out all 24 elements of G and write out how they multiply with each other. (I am assuming not many of you want to write out a 24-by-24 table.)

Theorem 26.3.8 is a promise that we can understand all the group multiplication in G simply by applying the group isomorphism, and converting the multiplication in G to multiplication in S_4.

And multiplication in S_4 is easy! Using cycle notation, we have become quite comfortable with composing bijections.

For sake of having precise discussions, let's set some notation by giving our diagonals names.

Notation 26.4.2 (The diagonals D_i)**.** Choose a face A of the cube once and for all. Then, choose a numbering $1, 2, 3, 4$ of the vertices, as before choosing the ordering to be counter-clockwise when the face is viewed from outside the cube.

For each i, we let D_i denote the diagonal passing through the vertex i and its opposite.

[1]For S_4, we already know this, as S_n has order $n!$. For G, the symmetries of the cube, this is Proposition 26.2.4.

Notation 26.4.3. We let $\rho : G \to S_4$ denote the group isomorphism from Theorem 26.3.8; it is defined in (26.3.0.1).

Example 26.4.4 (Rotations about an axis orthogonal to one face)**.** Let A be the face we chose in Notation 26.4.2. We let R_A be 90-degree rotation of the cube defined as follows: The axis of rotation is perpendicular to A and passes through the center of A, while we demand that the rotation is 90-degrees counter-clockwise when viewed from outside the cube toward A.

This rotation R_A sends the vertex 1 to 2, the vertex 2 to 3, and so forth; thus it has the following effect on diagonals:

$$D_1 \mapsto D_2, \qquad D_2 \mapsto D_3, \qquad D_3 \mapsto D_4, \qquad D_4 \mapsto D_1.$$

In other words,

$$\rho(R_A) = (1234).$$

Knowing that ρ is a group homomorphism (and doing no further geometry), we can then deduce

$$\rho(R_A^2) = (13)(24), \qquad \rho(R_A^3) = (1432).$$

In words: If we rotate the cube by 180 degrees about the axis normal to A, this has the effect of swapping the diagonal D_1 with the diagonal D_3, and D_2 with D_4. If we rotate by 270 degrees counterclockwise (that is, 90 degrees clockwise), then we cycle the diagonals by sending D_1 to D_4, D_4 to D_3, and so forth.

It is straightforward to check these claims geometrically, too, and not just take them as a consequence of composition in S_4.

Example 26.4.5. Now let B be the face of the cube which, when viewed from outside the cube, has vertices $1, 2, 4', 3'$ in counter-clockwise order (where $4'$ is the vertex diagonally opposite 4).[2]

Let R_B denote 90-degree rotation, counter-clockwise as viewed from outside the cube, about the axis normal to B. Then R_B acts as follows on vertices:

$$1 \mapsto 2, \qquad 2 \mapsto 4', \qquad 4' \mapsto 3', \qquad 3' \mapsto 1$$

and hence its effect on Diagonals is:

$$D_1 \mapsto D_2, \qquad D_2 \mapsto D_4, \qquad D_4 \mapsto D_3, \qquad D_3 \mapsto D_1.$$

[2]Note we are still using the numbering of vertices from Notation 26.4.2.

In other words,

$$\rho(R_B) = (1243).$$

Knowing that ρ is a group homomorphism (and doing no further geometry), we can then deduce

$$\rho(R_B^2) = (14)(23), \qquad \rho(R_B^3) = (1342).$$

Example 26.4.6. It is not so obvious how to geometrically interpret $R_B R_A$ – the rotation obtained by first doing R_A, and then applying R_B. But what this composition does to diagonals is easy to deduce by using composition in S_4:

$$\rho(R_B R_A) = \rho(R_B)\rho(R_A) = (1243)(1234) = (142).$$

So, amazingly, whatever $R_B R_A$ is, it is a rotation that *fixes* the diagonal D_3. (This means $R_B R_A$ will send the vertex 3 to itself and $3'$ to itself—in which case it is some rotation about D_3—or swap the vertices 3 and $3'$.)

Let me say something non-trivial here. I promise that you can now study the cube to name the unique rotation that acts by (142) on the set of diagonals. However, this requires work! Why? It's because the proof of surjection of ρ was formal—we just used that G and S_4 had the same cardinality and that ρ is injective. In particular, we never had to prove by hand that ρ is surjective, meaning we never had to take a permutation in S_4 and tell ourselves which rotation of the cube gives rise to this permutation.

Thus, the geometry of understanding which rotations induce permutations like (142) is precisely the ingredient one would use to prove by hand that ρ is a surjection.

In case you are curious, $R_B R_A$ acts as follows on the vertices:

$$(14'2')(21'4)(33')$$