Loose Ends; 2013.09.09

Lemma: $f: A \to B$ induces maps
$$f_n: H_nA \to H_nB$$

Pf If $\partial_n a = 0$, then $\partial_n^B f(a) = f\partial_n^A a = 0$. So $f(a) \in \operatorname{Ker} \partial_n^A$. So we have a map $\operatorname{Ker}\partial_n^A \to \operatorname{Ker}\partial_n^B$. Now define
$$f_n: [a] \longmapsto [f(a)].$$
If $a' = a + \partial_{n+1}^A \alpha$, then
$$f(a') = f(a) + \partial(f(\alpha))$$
so $[f(a')] = [f(a)]$. //.

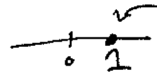
Last time.

$$\begin{array}{c} X_1 \\ \downdownarrows \partial \\ X_0 \end{array} \rightsquigarrow Ch(X) = \begin{array}{c} \mathbb{Z}X_1 \\ \downarrow \partial \\ \mathbb{Z}X_0 \end{array}.$$

Today: Simplicial sets.

Simplices

Defn The n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$ is the set ~~$\Delta^n \subset \mathbb{R}^{n+1}$~~
$$\left\{ (x_0, \dots, x_n) \;\middle|\; \begin{array}{l} \Sigma x_i = 1 \\ x_i \in [0,1] \end{array} \right\}.$$

Ex $\Delta^0 =$  point

$\Delta^1 =$  edge

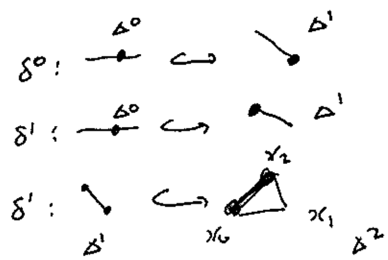
$\Delta^2 =$  triangle

Note that this definition gives nice way to phrase the faces of a simplex.

$\exists$ exactly $(n+1)$ injective maps
$$\delta^i: \{0, \dots, n\} \hookrightarrow \{0, \dots, n+1\}$$
that preserve the order of the elements — $0 < 1 < \dots < n$. Write δ^i for the map skipping the ith element.

Ex.
$$\delta^2: \begin{array}{ccc} 0 & \longrightarrow & 0 \\ 1 & & 1 \\ & & 2 \\ 2 & \longrightarrow & 3 \\ 3 & \longrightarrow & 4 \\ 4 & \longrightarrow & 5 \end{array}$$

Each δ^i induces a map
$$\Delta^n \hookrightarrow \Delta^{n+1}$$
by sending
$$x_j \longmapsto x_{\delta^i(j)}$$
and extending linearly.

Ex. 

Likewise, $\exists$ exactly n surjections
$$\sigma^i: \{0, \dots, n\} \to \{0, \dots, n-1\}, \quad 0 \le i \le n-1$$
that preserve order. σ^i is the map for which the preimage of i has cardinality two.

The observations so far suggest that the combinatorics of <u>simplices</u> is controlled by the combinatorics of totally ordered, finite sets.

~~Remark.~~

<u>Defn</u> Let Δ_{inj} be the category where

- objects are finite, non-empty, linearly ordered sets.
- morphisms are injective maps of ordered sets.

What does this category look like?

- Every object is isomorphic to the ordered set
  $$\{0, 1, \dots, n\} \quad i < i+1$$
  for some $n \in \mathbb{Z}_{\geq 0}$.

- Any map
  $$[n] \longrightarrow [n'], \quad n' > n$$
  factors as a sequence of δ^i.

So for convenience, we can "draw" this category by only drawing morphisms between objects of successive cardinality:

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "\delta^0"] \arrow[r, shift right, "\delta^1"'] & {[1]} \arrow[r, bend left, "\delta^0"] \arrow[r, "\delta^1"] \arrow[r, bend right, "\delta^2"'] & {[2]} \arrow[r, bend left, "\delta^0"] \arrow[r, shift left, "\delta^1"] \arrow[r, shift right, "\delta^2"'] \arrow[r, bend right, "\delta^3"'] & {[3]} \cdots \end{tikzcd}$$

<u>Exercise</u>.

If $i < j$, then
$$\delta^j \delta^i = \delta^i \delta^{j-1}.$$

<u>Pf</u> Next time.

~~Prop.~~

<u>Defn</u>. Let
$$\Delta : \Delta_{inj} \longrightarrow \text{Spaces}$$
be the functor which sends
$$[n] \longmapsto \Delta^n$$
$$(f : [n] \to [n']) \longmapsto \begin{array}{l} \Delta^n \longrightarrow \Delta^{n'} \\ x_i \longmapsto x_{f(i)}, \end{array} \text{ extend linearly.}$$

So Δ defines a diagram of spaces
$$\Delta^0 \rightrightarrows \Delta^1 \Rrightarrow \Delta^2 \Rrightarrow \Delta^3 \cdots$$
where each arrow is the inclusion of some ~~space~~ face $\Delta^i \hookrightarrow \Delta^{i+1}$.

Last time I said that if a space X is glued together from simplices X_i, then we should expect a diagram of the shape
$$X_0 \leftleftarrows X_1 \Lleftarrow X_2 \Lleftarrow X_3 \cdots$$
where all the arrows are <u>reversed</u>! If you've done the exercise, you might know the terminology for reversing arrows.

Defn A <u>semisimplicial set</u> is a functor

$$X_\bullet : (\Delta_{inj})^{op} \longrightarrow \text{Sets}.$$

A <u>simplicial set</u> is a functor

$$\Delta^{op} \longrightarrow \text{Sets}.$$

Δ has same objects as Δ_{inj}, but allows <u>all</u> order-preserving maps.

Since $X_\bullet$ is contravariant, maps

$$\delta^i : [n] \longrightarrow [n+1]$$

become maps

$$d_i : X_{n+1} \longrightarrow X_n$$

<u>Exercise</u> Show that if $i<j$, $d_i d_j = d_{j-1} d_i$.

Defn If $X_\bullet$ is a (semi) simplicial set, the <u>simplicial chain complex</u> of $X_\bullet$ is defined to be the chain complex

$$Ch(X)_n := \mathbb{Z}X_n,$$

$$\partial_n := \sum_{0\le i\le n} (-1)^i d_i .$$

<u>Rmk</u> This defn depends only on d_i, so doesn't matter whether $X_\bullet$ is (semi)simplicial.

<u>Ex.</u> If $X_\bullet$ is a directed graph

$$X_0 \leftleftarrows X_1 \leftarrow \emptyset \leftarrow \emptyset \cdots$$

we have a chain complex

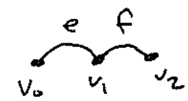
$$0 \leftarrow \mathbb{Z}X_0 \xleftarrow{\partial} \mathbb{Z}X_1 \leftarrow 0 \leftarrow 0 \cdots$$

with $\partial(e) = d_0(e) - d_1(e)$.

<u>Exer</u> Compute simplicial homology (ie, the homology groups $\mathrm{Ker}\,\partial_n / \mathrm{Im}\,\partial_{n+1} = H_n$ for $Ch(X_\bullet)$) for

(a) $X =$ 
n vertices, no edges

(b) $X =$ 

(c) $X =$ 

(d) $X =$  one vertex, one edge

(e) $X =$ 

(f) $X =$ 

Defn. If $X_\bullet$ is a semisimplicial set, its <u>geometric realization</u> is the space

$$|X_\bullet| := \coprod_n X_n \times \Delta^n \Big/ (d_i x, \vec{u}) \sim (x, \delta^i \vec{u}).$$

If $X_\bullet$ is a simplicial set, we define its geometric realization to be

$$|X_\bullet| := \coprod_n X_n \times \Delta^n \Big/ \begin{array}{l}(d_i x, \vec{u}) \sim (x, \delta^i \vec{u}) \\ (s_i x, \vec{u}) \sim (x, \sigma^i \vec{u}).\end{array}$$

As you might imagine, given a space X, there may be a ton of ways to find a semisimplicial set $X_\bullet$ for which
$$|X_\bullet| \cong X.$$

But there is a very natural one you can extract:

<u>Defn</u>. Given a space X, let $\mathrm{Sing}(X)_n$ be the set of continuous maps
$$\Delta^n \longrightarrow X.$$

Since
$$\Delta : \mathbb{\Delta} \to \mathrm{Spaces}, \quad [n] \mapsto \Delta^n$$
is a covariant functor, $\mathrm{Sing}(X)_\bullet$ defines a contravariant functor.
$$[n] \xmapsto{\Delta} \Delta^n \xmapsto{\hom_{\mathrm{Spaces}}(\bullet, X)} \hom_{\mathrm{Spaces}}(\Delta^n, X)$$
$$([n] \to [n']) \xmapsto{\Delta} (\Delta^n \to \Delta^{n'}) \xmapsto{\hom_{\mathrm{Spaces}}(-, X)} \left(\hom_{\mathrm{Spaces}}(\Delta^n, X) \leftarrow \hom(\Delta^{n'}, X)\right).$$

<u>Defn</u> $\mathrm{Sing}(X)$ is called the simplicial set of <u>singular chains</u> of X. Each element in $\mathrm{Sing}(X)_n$ is called a singular n-chain of X.

$$\begin{array}{c}
\vdots \quad \uparrow\uparrow\uparrow s \\
\mathrm{Maps}(\Delta^2, X) \\
d \downarrow\downarrow\downarrow \quad \uparrow\uparrow s \\
\mathrm{Maps}(\Delta^1, X) \\
d \downarrow\downarrow \quad \uparrow s \\
\mathrm{Maps}(\Delta^0, X)
\end{array}$$

$(\mathrm{Maps} := \mathrm{Hom}_{\mathrm{Spaces}})$

<u>Defn</u> Given a space X, the <u>singular chain complex</u> of X is $\mathrm{Ch}(\mathrm{Sing}(X)_\bullet)$.

The homology groups of this chain complex are called the <u>homology groups</u> of X.

<u>Propn</u> Any continuous map
$$f : X \longrightarrow Y$$
induces a map of chain complexes
$$\mathrm{Ch}(\mathrm{Sing}(X)) \longrightarrow \mathrm{Ch}(\mathrm{Sing}(Y)).$$

Big picture:

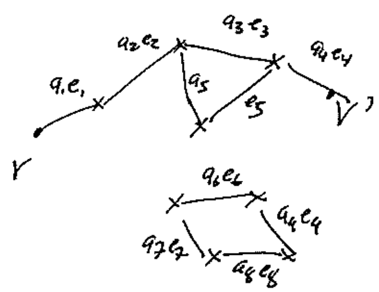
$$\begin{array}{ccccccc} \text{Spaces} & \xrightarrow{\text{Sing}} & \text{sSets} & \xrightarrow{\text{Ch}} & \text{Chain} & \xrightarrow{H_i} & \text{Ab} \\ X & \longmapsto & \text{Sing}(X_\bullet) & \longmapsto & \text{Ch}(\text{Sing}(X_\bullet)) & \longmapsto & H_i(X) \end{array}$$

(1) Geometric interpretation

(2) Computability.

(1) H_0: Every chain $\sum a_i v_i \in \text{Ch}(\text{Sing})_0$ is closed, since $\partial_0 = 0$.

Further, note $v \sim v'$ iff v and v' are endpoints of some path $e \in \text{Sing}(X)_1$. (If $v - v' = \partial \sum a_i e_i$, just concatenate the paths; argument needs to address multiplicity, but not hard.)



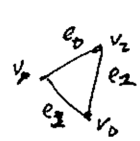
Prop'n $\exists$ isomorphism

$$H_0(X) \cong \mathbb{Z}(\pi_0 X).$$

So rank $H_0(X)$ is # of conn. components of X.

(1) H_1: When is some $\alpha = \sum a_i e_i \in \text{Ch}(\text{Sing}(X))_1$ closed?

Ex

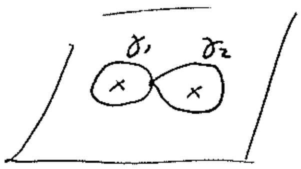


$$\partial(e_0 + e_1 + e_2) = \underbrace{v_2 - v_1}_{\partial e_0} + \underbrace{v_1 - v_0}_{\partial e_2} + \underbrace{v_0 - v_2}_{\partial e_1} = 0.$$

In general, α looks like some "sum" of closed paths.

And $\partial\beta = \alpha \iff \alpha$ is "boundary" of some surface made out of triangles.

Ex. $X = \mathbb{R}^2 \setminus \{0,1\}$



Then γ_i above will define two elements of $H_1(X)$. But $[\gamma_1]+[\gamma_2] = [\gamma_2]+[\gamma_1] \in H_1$, while in $\pi_1(X, x_0)$, $\gamma_1\gamma_2 \neq \gamma_2\gamma_1$.

We see seeds for

Thm If X is path-connected, then

$$H_1(X, \mathbb{Z}) \cong \text{abelianization}(\pi_1(X, x_0)).$$

Further, this $\cong$ is natural, so $\forall$ $f: X \to Y$ continuous, the following diagram of groups commutes:

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r] \arrow[d] & \pi_1(Y, f(x_0)) \arrow[d] \\
H_1(X) \arrow[r] & H_1(Y)
\end{tikzcd}$$

(1) H_n, $n \geq 2$: More generally, n^{th} homology counts n-dimensional objects without boundary, modulo the relation that two such things are identified if an $(n+1)$-dimensional object interpolates from one to the other. ("object" is purposefully a vague word; whether these can be represented by manifolds is a delicate question.)