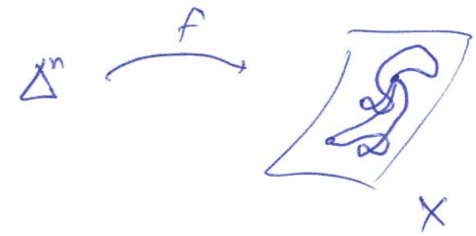
Summary of what we've done so far:

To every topological space X, we've assigned a chain complex called the <u>complex of singular chains</u>, or <u>singular chain complex</u> of X.

because an arbitrary map

$$\Delta^n \longrightarrow X$$

can look very singular!



$$\begin{tikzcd}
\vdots \arrow[d] \\
\mathbb{Z}\,\mathrm{Maps}(\Delta^2, X) \arrow[d, "\partial_2"] \\
\mathbb{Z}\,\mathrm{Maps}(\Delta^1, X) \arrow[d, "\partial_1"] \\
\mathbb{Z}\,\mathrm{Maps}(\Delta^0, X) \arrow[d] \\
0 \arrow[d] \\
\vdots
\end{tikzcd}$$

continuous maps from Δ^2 to X

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i$$

<u>Good</u>: This gives a <u>functor</u> Spaces $\longrightarrow$ Chain Complexes. (Homework!)

<u>Bad</u>: This is hard to compute just from the definitions.

<u>Notation</u>: $\mathbb{Z}\,\mathrm{Maps}(\Delta^n, X) =: C_n(X)$.

$$\bigoplus_{i \in \mathbb{Z}} H_i(X) =: H_*(X)$$

Example If $X = \{*\}$, what is $H_\bullet(X)$?

Singular chain complex is

$$\begin{array}{ccc} & & \vdots \\ \downarrow & & \downarrow \\ \mathbb{Z}\,\mathrm{Maps}(\Delta^n, X) & & \mathbb{Z} \\ \downarrow & = & \downarrow \\ \mathbb{Z}\,\mathrm{Maps}(\Delta^{n-1}, X) & & \mathbb{Z} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

Since $\mathrm{Maps}(\text{blah}, \text{point}) = \text{point}$. (Every map is the constant map.)

and $\partial_n(a_n) = \sum_{i=0}^{n} (-1)^i d_i(a_n) = a_{n-1} - a_{n-1} + a_{n-1} - \cdots \pm a_{n-1}$

$$= \begin{cases} 0 & \text{if } n \text{ odd} \\ a_{n-1} & \text{if } n \text{ even}. \end{cases}$$

So we have

$$\begin{array}{cc} & \vdots \\ & \downarrow \times 1 \\ 3 & \mathbb{Z} \\ & \downarrow 0 \\ 2 & \mathbb{Z} \\ & \downarrow \times 1 \\ 1 & \mathbb{Z} \\ & \downarrow 0 \\ 0 & \mathbb{Z} \\ & \downarrow \\ & 0 \end{array} \qquad \Longrightarrow \qquad \begin{array}{l} \vdots \\ H_3 \cong \mathbb{Z}/\mathbb{Z} \cong 0 \\ H_2 \cong 0/0 \cong 0 \\ H_1 \cong \mathbb{Z}/\mathbb{Z} \cong 0 \\ H_0 \cong \mathbb{Z} \end{array}$$

So $H_*(pt) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise}. \end{cases}$

If $X = S^1$, or something even simpler, $\mathrm{Maps}(\Delta^n, X)$ is a huge set.

The reason we find a semisimplicial set $X_\bullet$ s.t.

$$|X_\bullet| \cong X$$

is to compute $H_*(X)$ using a smaller, easier chain complex.

So we'll need to prove

Thm If $X_\bullet$ is a semisimplicial set,
and X is a space
such that

$$|X_\bullet| \cong X \quad (\text{homeomorphism})$$

then the simplicial homology of $X_\bullet$ is isomorphic to the singular homology of X. i.e.,

$$H_*(Ch(X_\bullet)) \cong H_*(X) \quad \forall * \in \mathbb{Z}.$$

Idea: Given X , maps $\Delta^n \to$  can look crazy.
A homeomorphism $|X_\bullet| \to X$ picks out maps that comply w/ a chosen simplicial decomposition of X.

, force Δ^n to land as some simplex of $|X_\bullet|$.

But the bigness and flimsiness of <u>singular chains</u> can help prove more structural results about the property of $H_0(X)$. We'll exploit that later.

For now let's get some intuition.

<u>Defn</u>. We say a space X is path-connected if $\forall\ x_0, x_1 \in X$, $\exists$ a continuous map
$$\gamma : [0,1] \longrightarrow X$$
s.t. $\gamma(0) = x_0$, $\gamma(1) = x_1$.

<u>Propn</u>. ① The map
$$\varepsilon : \mathbb{Z}\,Map(\Delta^0, X) \longrightarrow \mathbb{Z}$$
$$\sum a_i f_i \longmapsto \sum a_i$$
is a group homomorphism.

<u>Propn</u> ② $\operatorname{Ker} \varepsilon = \operatorname{Im} \partial_1$,

if X is path-connected, and NOT empty.

<u>Cor</u>. If X is path-connected, and non-empty,
$H_0(X) \cong \mathbb{Z}$.

<u>Pf of Corollary</u>: If X is non-empty, ε is a surjection,

so $\quad \mathbb{Z} \cong \mathbb{Z}\,Map(\Delta^0, X) / \operatorname{Ker} \varepsilon$.

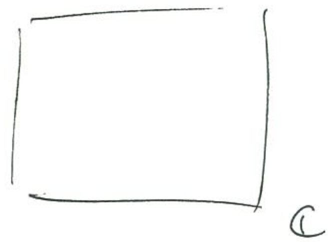
Since $\partial_0 \equiv 0$ by definition, $\quad \mathbb{Z}\,Map(\Delta^0, X) = \operatorname{Ker} \partial_0$
and by Propn ②, $\quad \operatorname{Ker} \varepsilon = \operatorname{Im} \partial_1$.

Hence
$$\mathbb{Z} \cong \operatorname{Ker}\partial_0 / \operatorname{Im}\partial_1 =: H_0(X).$$

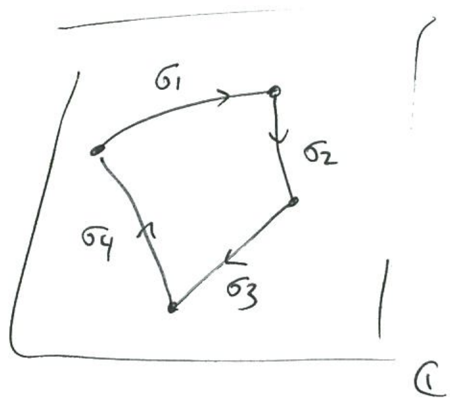
So what about H_1? No huge theorems yet; just examples.

Examples

If $X = \mathbb{C}$,

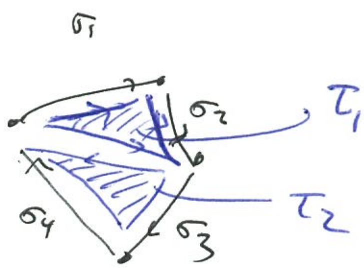


a typical closed element $\sigma = \sum a_i \sigma_i \in C_1(X)$, $\sigma_i : [0,1] \to X$ might look like



(a closed path, broken into line segments)

If this is in turn exact so $\sigma = \partial\tau$ for some $\tau \in C_2$, τ might look like



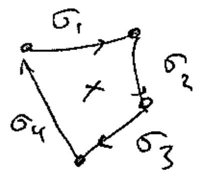
$\tau = \tau_1 + \tau_2$.

In other words, σ is exact if some "surface" τ realizes σ as the boundary of τ. In this pictorial example,

$$[\sigma] = [\partial\tau], \quad \text{so} \quad [\sigma] = 0 \in H_1(X).$$

Now, if you were to puncture $\mathbb{C}$, so $X = \mathbb{C} \setminus \{0\}$,
then no surface could "fill in" the interior of σ:

I'm not asserting a proof yet, but this is the gyst.



so for this X, σ is a closed cycle which is NOT exact.

Lesson: H_1 "counts" holes in X.

This interpretation won't always be sensible; we'll find spaces where $H_1 \cong \mathbb{Z}/2\mathbb{Z}$, for instance.

Now let's talk about some properties of C_* and H_*.

Thm (Mayer-Vietoris)
Let $U \cup V = X$, w/ $U, V \subset X$ open. Then you can compute $H_\bullet(X)$ by knowing

- $H_\bullet(U)$
- $H_\bullet(V)$
- $H_\bullet(U \cap V)$
- how $U \cap V$ embeds into U and V.

This is some local-to-global process: If you understand $H_\bullet$ on small pieces of X, you can understand it on all of X. Very convenient. I'll give a ~~mope~~ more precise statement next time.

<u>Defn</u> Let $f_0, f_1 : X \longrightarrow Y$ be two continuous maps.

We say <u>f_0 is homotopic to f_1</u> if $\exists$ a continuous map

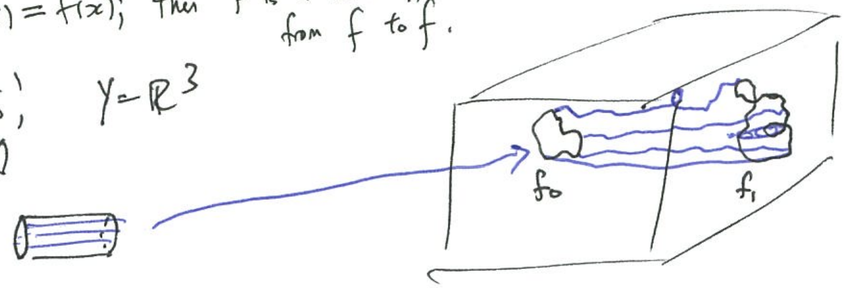
$$F: X \times [0,1] \longrightarrow Y$$

such that
$$F|_{X \times \{0\}} = f_0, \qquad F|_{X \times \{1\}} = f_1.$$

<u>Ex</u> Every map is homotopic to itself!
Set $F(x,t) = f(x)$; then F is a homotopy from f to f.

<u>Ex</u> $X = S^1$; $Y = \mathbb{R}^3$

$X \times [0,1] =$



A picture of f_0 "wiggling" into f_1.

Now, by the functor $C_* : \text{Spaces} \longrightarrow \text{Chain complexes}$,
f_0 and f_1 induce map of chain complexes.

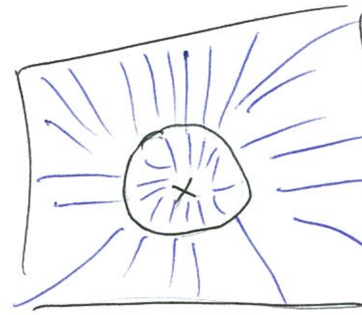
$$C_*(X) \underset{(f_1)_*}{\overset{(f_0)_*}{\rightrightarrows}} C_*(Y)$$

And inturn, on homology:

$$H_*(X) \underset{(f_1)_*}{\overset{(f_0)_*}{\rightrightarrows}} H_*(Y).$$

<u>Thm</u> If f_0 and f_1 are homotopic,
they induce the same map on homology.

For the time being, these two theorems will buy us a lot of computational mileage. Before proving them, we'll look at applications.



<u>Ex.</u> Consider the inclusion

$$\begin{aligned} S^1 &\xrightarrow{i} \mathbb{C}^\times \\ \theta &\longmapsto e^{i\theta}. \end{aligned}$$

There's a projection map

$$\begin{aligned} \mathbb{C}^\times &\xrightarrow{p} S^1 \\ r\cdot e^{i\theta} &\longmapsto e^{i\theta} \end{aligned}$$

~~so~~ where $p \circ i = id_{S^1}$.

This means the composition

$$H_*(S^1) \xrightarrow{i_*} H_*(\mathbb{C}) \xrightarrow{p_*} H_*(S^1)$$

is equal to the map

$$H_*(S^1) \xrightarrow{id} H_*(S^1).$$

Functoriality.
$\forall j$, H_j: Spaces $\to$ AbGps is a functor, so
$(p \circ i) = id$
$\Rightarrow (p\circ i)_* = id_{H_j}$ (by functor sending id to id)
and $(p \circ i)_* = p_* \circ i_*$ (by functor respecting composition)

Also consider the map

$$\begin{aligned} F: \mathbb{C}^\times \times [0,1] &\longrightarrow \mathbb{C}^\times \\ (re^{i\theta}, t) &\longmapsto e^{i\theta} + t(r-1)e^{i\theta}. \end{aligned}$$

Then

$$\begin{aligned} F|_{\mathbb{C}^\times \times \{0\}} : \mathbb{C}^\times &\longrightarrow \mathbb{C}^\times \\ re^{i\theta} &\longmapsto e^{i\theta} = i\circ p \end{aligned}$$

and

$$\begin{aligned} F|_{\mathbb{C}^\times \times \{1\}} : \mathbb{C}^\times &\longrightarrow \mathbb{C}^\times \\ re^{i\theta} &\longmapsto re^{i\theta} = id_{\mathbb{C}^\times}. \end{aligned}$$

In other words, $i \circ p$ is <u>homotopic</u> to $id_{\mathbb{C}^\times}$!

This means $(i \circ p)_* = (id_{\mathbb{C}^\times})_*$ by theorem. So

$$\begin{tikzcd} H_*(S^1) \arrow[r, "i_*"] \arrow[dr, "id"'] & H_*(\mathbb{C}^\times) \arrow[d, "p_*"] \arrow[dr, "id"] & \\ & H_*(S^1) \arrow[r, "i_*"'] & H_*(\mathbb{C}^\times) \end{tikzcd}$$

In other words, both i_* and p_* are $\cong$ on H_*.

$\Longrightarrow \quad H_*(S^1) \cong H_*(\mathbb{C}^\times)$!

Let's extract the essence of this situation.

<u>Defn</u> Let $f: X \to Y$ be a continuous map.
We say f is a <u>homotopy</u> <u>equivalence</u> if
$\exists$ $\underbrace{\text{a map}}^{\text{continuous}}$ $g: Y \to X$ such that

- $g \circ f$ is homotopic to id_X
- $f \circ g$ is homotopic to id_Y.

<u>Rmk</u> If "is homotopic to" is replaced by "is equal to," this would be definition of <u>homeomorphism</u>. Homeomorphisms are special examples of homotopy equivalences.

<u>Defn</u> Two spaces are called <u>homotopy</u> <u>equivalent</u> if $\exists$ a homotopy equivalence between them.

Example S^1 and $\mathbb{C}^\times$ are homotopy equivalent, as we've shown.

Prop'n If X and Y are homotopy equivalent, then their homology groups are isomorphic.

Now let's prove the theorem about homotopies.

Def'n Let $f, g : A_\cdot \to B_\cdot$ be ~~ch~~ maps of chain complexes. A chain homotopy, or homotopy, between f and g B

- A sequence of maps
$$F_n : A_n \longrightarrow B_{n+1} \quad \text{(note the degree shift!)}$$

such that

- $\forall n, \quad f_n - g_n = \partial^B_{n+1} F_n + F_{n-1} \partial^A_n$.

$$\begin{tikzcd}
 & B_{n+1} \arrow[d, "\partial_n"] \\
A_n \arrow[ur, "\pm F_n"] \arrow[r, "f_n - g_n"] \arrow[d, "\partial_n"'] & B_n \\
A_{n-1} \arrow[ur, "\mp F_{n-1}"'] &
\end{tikzcd}$$

diagram commutes up to $f-g$.

Prop'n If $\exists$ a chain homotopy between f and g, then f and g define the same maps on homology:
$$g_* = f_* : H_\cdot(A) \longrightarrow H_\cdot(B)$$

Pf. $[f(a)] - [g(a)] = [f(a) - g(a)]$

$= [\partial F(a) + F\partial(a)]$

$= [\partial F(a)] + [F\partial(a)]$

$= 0 - 0$

since $\partial F(a)$ is exact

since $a \in \mathrm{Ker}\, \partial^A$. //

Lemma Let f, g be continuos maps $X \to Y$.

A homotopy $F : X \times [0,1] \longrightarrow Y$

induces a chain homotopy between the maps

$$C_*(f) : C_*(X) \longrightarrow C_*(Y)$$

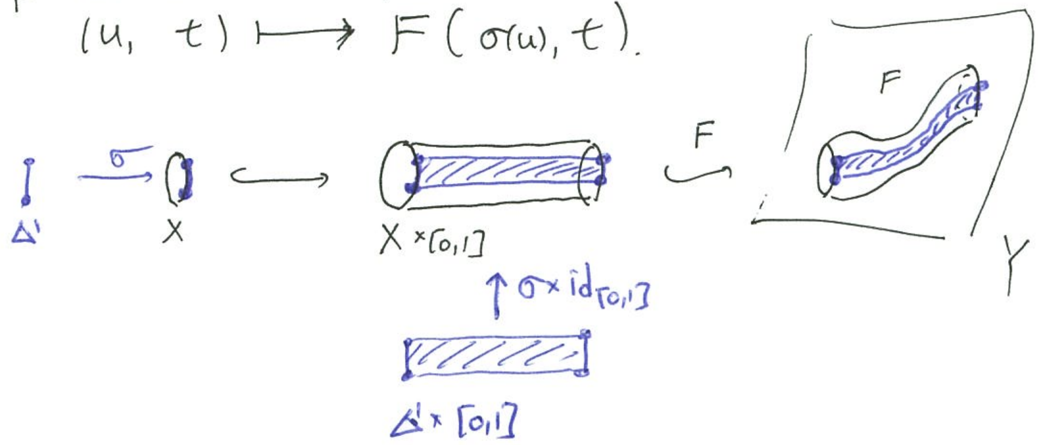
and

$$C_*(g) : C_*(X) \longrightarrow C_*(Y).$$

Pf. Given $F: X\times[0,1] \longrightarrow Y$,

every map $\sigma: \Delta^n \longrightarrow X$ defines a map

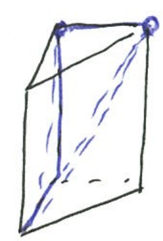
$$\sigma_F : \Delta^n \times [0,1] \longrightarrow Y$$
$$(u, t) \longmapsto F(\sigma(u), t).$$

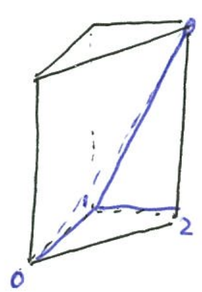


We give a simplicial structure on $\Delta^n\times[0,1]$

$$\begin{tikzcd}[row sep=small] (0,1) \arrow[d, dash] \\ (0,0) \end{tikzcd}$$

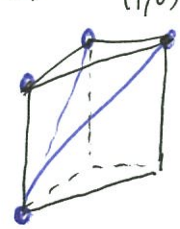
$$\begin{tikzcd}[row sep=small] (0,1) & (1,1) \arrow[d, dash] \\ (0,0) \arrow[r, dash] & (1,0) \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] (0,1) \arrow[r, dash] \arrow[d, dash] & (1,1) \\ (0,0) & (1,0) \end{tikzcd}$$





$$\begin{tikzcd}[row sep=small] (0,1) & (1,1) & (2,1) \arrow[d, dash] \\ (0,0) \arrow[r, dash] & (1,0) \arrow[r, dash] & (2,0) \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] (0,1) & (1,1) \arrow[r, dash] \arrow[d, dash] & (2,1) \\ (0,0) \arrow[r, dash] & (1,0) & (2,0) \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] (0,1) \arrow[r, dash] \arrow[d, dash] & (1,1) \arrow[r, dash] & (2,1) \\ (0,0) & (1,0) & (2,0) \end{tikzcd}$$



The space $\Delta^n \times [0,1]$ has vertices of the form

$$(i,j) \qquad \begin{array}{l} i \in \{0,\dots,n\} \\ j \in \{0,1\} \end{array}.$$

Explicitly, (i,j) is the point $(x_i = 1, j) \in \Delta^n \times [0,1]$.

$\forall$ $k \in \{0,\dots,n\}$, the convex hull of the vertices

$$(0,0),\ (1,0),\ (2,0),\ \dots,\ (k,0),\ (k,1),\ (k+1,1),\ \dots,\ (n,1)$$

forms an $(n+1)$-simplex. $\Delta^n \times [0,1]$ is the union of all these simplices, $0 \le k \le n$.

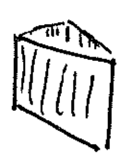
Call them σ_k. Then the composite

$$\Delta^{n+1} \xrightarrow{\sigma_k} \Delta^n \times [0,1] \xrightarrow{\sigma \times id_{[0,1]}} X \times [0,1] \xrightarrow{F} Y$$

defines a $(n+1)$-simplex $\alpha_k \in Sing(Y)_{n+1}$. Define

$$\begin{aligned} F_n : Sing(X)_n &\longrightarrow Sing(Y)_{n+1} \\ \sigma &\longmapsto \sum (-1)^k \alpha_k . \end{aligned}$$

Then $\partial F_n(\sigma)$ "="  = boundy of prism (top, bottom, walls included).

$F_{n-1}\partial(\sigma) =$  = walls of prism (no top no bottom)

So $\underset{C_*(f) - C_*(g).}{\underset{\parallel}{\partial F + F\partial}}$ (we'll work out the signs).

//