Proof of Homotopy Invariance.

Thm. If f_0, f_1 are homotopic maps from X to Y, $(f_0)_*$ and $(f_1)_*$ are the same map on homology.

How would we prove such a thing?

Well, f_0 and f_1 both induce maps at the level of singular chains — that is,

$$f_0 : C_\bullet(X) \to C_\bullet(Y), \quad f_1 : C_\bullet(X) \to C_\bullet(Y).$$

(I'm using f_i to denote both the continuous map and the chain map, be careful!)

So is there something we can say about these maps at the chain level?

Defn: Let f_0 and f_1 be maps of chain complexes,

$$f_0, f_1 : A \longrightarrow B.$$

A chain homotopy between f_0 and f_1 is the data of

of abelian groups

- A map $F_n : A_n \longrightarrow B_{n+1}$ $\forall\, n \in \mathbb{Z}$

such that

$$f_0 - f_1 = F\partial^A + \partial^B F.$$

shorthand for

$$(f_0)_n - (f_1)_n = F_{n-1}\partial_n^A + \partial_{n+1}^B F_n, \ \forall n.$$

The diagram of groups looks like this:

$$\begin{tikzcd}
\arrow[d] & \arrow[d] \\
A_{n+2} \arrow[d] & B_{n+2} \arrow[d, "\partial"] \\
A_{n+1} \arrow[d, "\partial"'] \arrow[ur, "F_{n+1}"] \arrow[r, phantom, "\boxed{f_0 - f_1}"] & B_{n+1} \arrow[d, "\partial"] \\
A_n \arrow[d, "\partial"'] \arrow[ur, "F_n"] \arrow[r, phantom, "\boxed{f_0 - f_1}"] & B_n \arrow[d] \\
A_{n-1} \arrow[d] \arrow[ur, "F_{n-1}"] & B_{n-1} \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

that is, this parallelogram is NOT commutative; it's commutative <u>up to</u> the difference of f_0 and f_1.

This seems abstract—what's the geometric intuition?

Well, if $A = C_\bullet(X)$, $B = C_\bullet(Y)$, and f_0, f_1 are the chain maps induced by continuous maps f_0, f_1, what would we expect F to be?

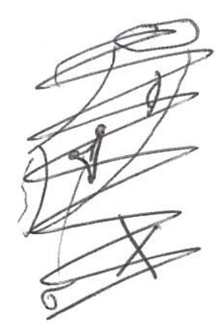
We'd expect it to be some chain map induced by the homotopy $F: X \times [0,1] \longrightarrow Y$.

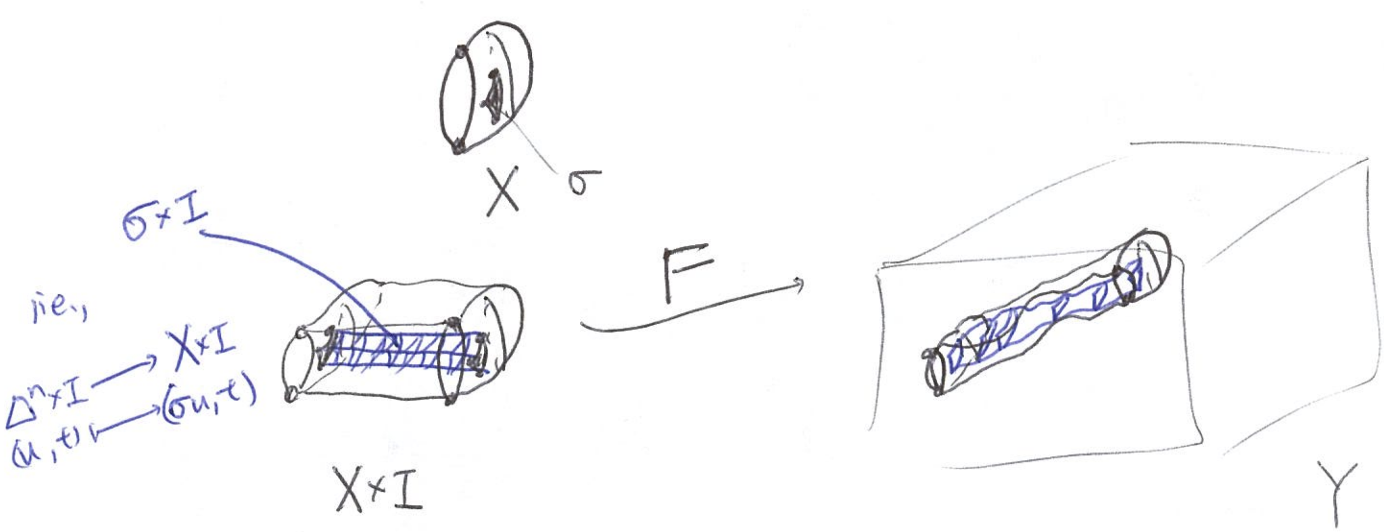
Well, any $\sigma: \Delta^n \longrightarrow X$

induces a map

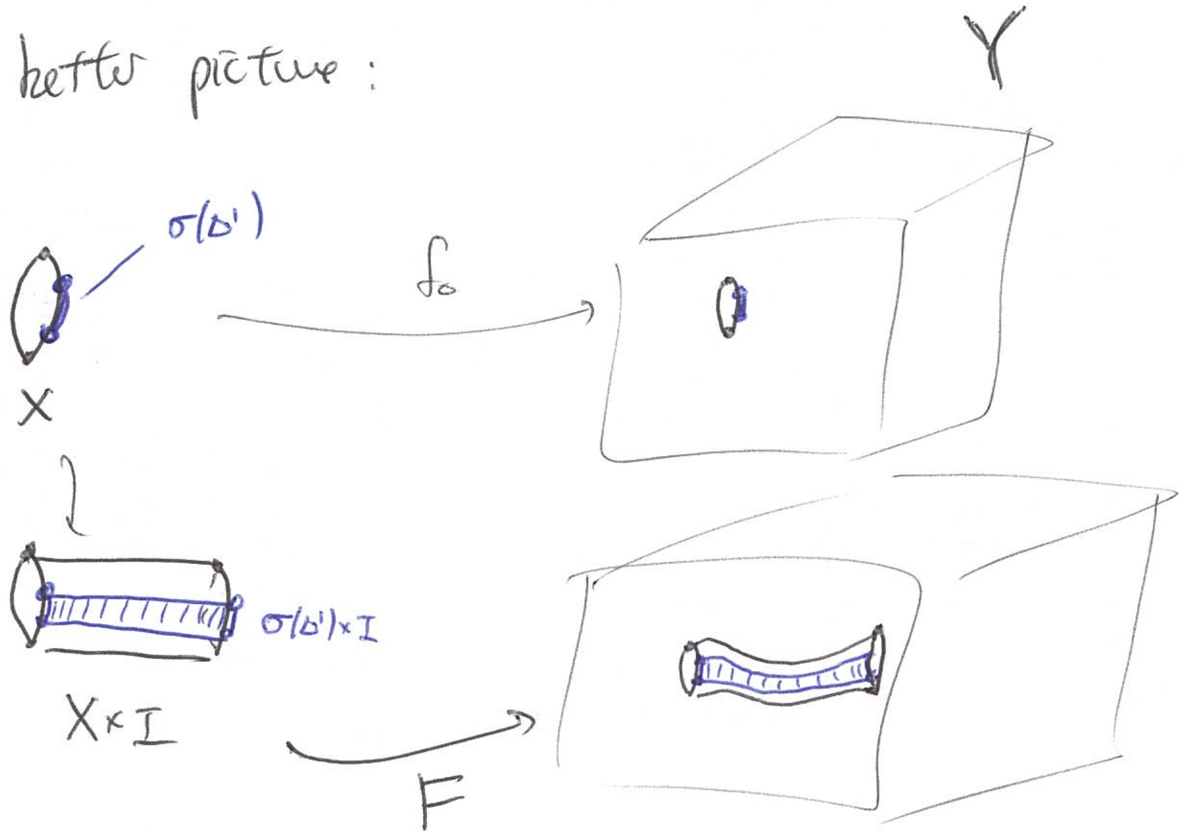
$$\begin{aligned} \Delta^n \times [0,1] &\xrightarrow{F\sigma} Y \\ (u, t) &\longmapsto F(\sigma(u), t). \end{aligned}$$

Pictorially,



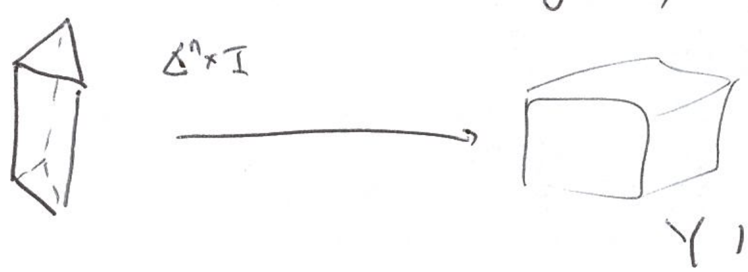


A better picture:

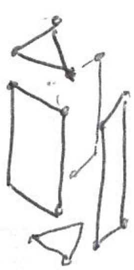


So what does $\partial F + F\partial$ mean?

If you believe F is the chain given by a map



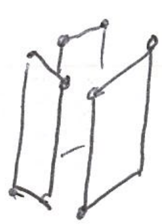
$\partial F =$



and $F\partial = F($

$)$

$=$



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No top+bottom, just the walls!

So you can imagine that the vertical walls of this prism cancel out, and you're left w/ just the top and bottom copies of Δ^n.

But $F|_{\Delta^n \times \{0\}} = f_0$ and $F|_{\Delta^n \times \{1\}} = f_1$.

So up to signs, we see a geometric intuition for why we want a formula like

$$\partial F + F\partial = f_0 - f_1.$$

<u>Moreover:</u>

<u>Prop'n.</u> If $f_0, f_1 : A \to B$ are chain maps that are chain homotopic, they induce the same maps on homology.

i.e., $\exists$ a chain homotopy btwn them.

$$(f_0)_* = (f_1)_* : H_n(A) \to H_n(B) \ \forall n.$$

Pf. Let $F : A_\circ \longrightarrow B_{\circ+1}$ be the chain homotopy,

so $(\partial F + F\partial)(a) = f_0(a) - f_1(a) \quad \forall\, a \in A_n, \ \forall n.$

Then

$$
\begin{aligned}
(f_0)_*[a] - (f_1)_*[a] &= [f_0 a] - [f_1 a] \\
&= [f_0 a - f_1 a] \\
&= [\partial F a + F \partial a] \\
&= [\partial F a] + [F \cdot 0] \\
&= [0]
\end{aligned}
$$

$\partial a = 0$ since $a \in \operatorname{Ker} \partial$ by def'n of homology

since we mod out by image(∂) by def'n of homology.

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So given $F: X \times [0,1] \longrightarrow Y$,

we need to produce maps

$$F_n: C_n(X) \longrightarrow C_{n+1}(Y).$$

The motivation is to send

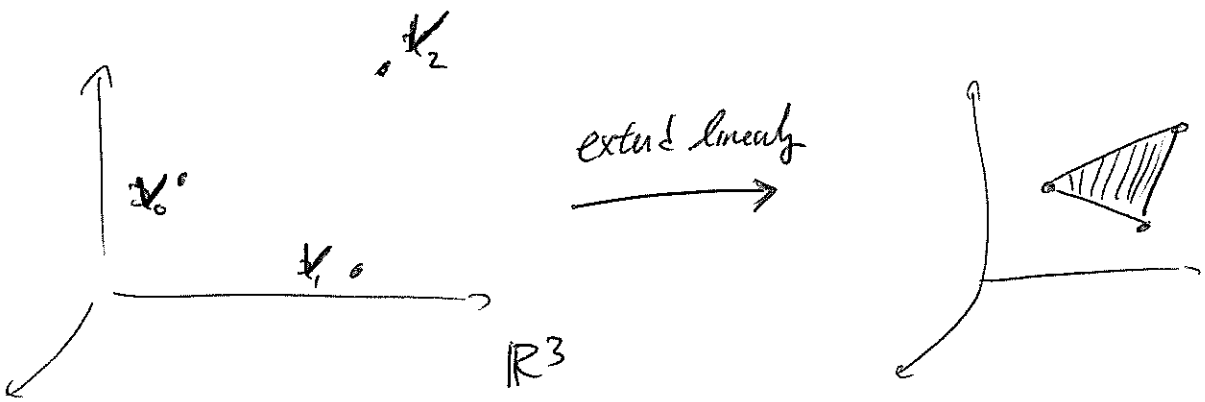
$$\sigma: \Delta^n \to X$$

to something that represents the induced map

$$\Delta^n \times [0,1] \longrightarrow Y.$$

Unfortunately, $\Delta^n \times [0,1]$ is <u>NOT</u> a simplex (unless $n=0$) so we want to give it ~~th~~ simplicial decomposition.

Observe that the data of $(n+2)$, ordered, unequal points in $\mathbb{R}^{n+2}$ is enough to determine a map $\Delta^{n+1} \longrightarrow \mathbb{R}^{n+2}$.



Explicitly, since $\Delta^{n+1} = \{(x_0, \ldots, x_{n+1}) \mid \sum x_i = 1,\ x_i \in [0,1]\}$ just take the affine linear map

$$j : \mathbb{R}^{n+2} \longrightarrow \mathbb{R}^{n+2}$$

$$\underset{\substack{\uparrow \\ i^{th} \text{ spot}}}{(0, \ldots, 0, 1, 0, \ldots, 0)} \longmapsto v_i .$$

We'll call the image the <u>convex hull</u> of the v_i.

ie, $\mathrm{hull}(\{v_i\}) := \mathrm{image}(j|\Delta^{n+1})$.

Now lets consider

$$\Delta^n \times [0,1] \subset \mathbb{R}^{n+1} \times \mathbb{R} \cong \mathbb{R}^{n+2}.$$

I'll give you $n+1$ copies of Δ^{n+1} ~~that~~ whose union is the prism $\Delta^n \times [0,1]$.

By previous remark, for each $a \in \{0,1,\cdots,n\}$ indexing the $n+1$ copies

I'll give you $n+2$ vertices $V_i^a \in \Delta^n \times [0,1] \subset \mathbb{R}^{n+1} \times \mathbb{R}$.

indexes the vertices of each copy.

And this defines a map

$$\tau^a : \Delta^{n+1} \longrightarrow \Delta^n \times [0,1] \quad \forall\, a \in \{0,1,\cdots,n\}.$$

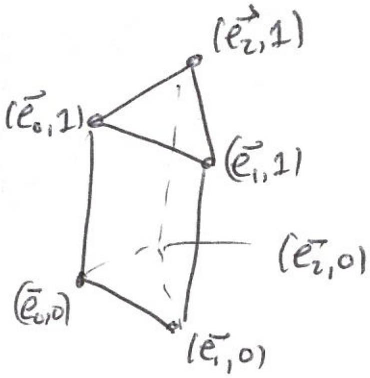
<u>Defn</u> Let

$$V_i^{\,a} = \begin{cases} (\vec{e}_i, 0) & i \le a \\ (\vec{e}_i, 1) & i \ge a+1 \end{cases}$$

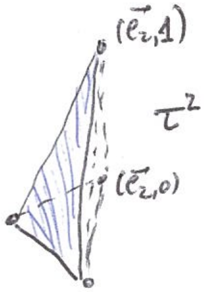
$a \in \{0, \dots, n\}$

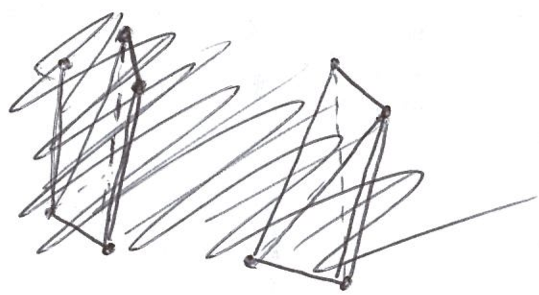
$i \in \{0, 1, \dots, n+1\}$.

<u>Ex</u> $n=2$.



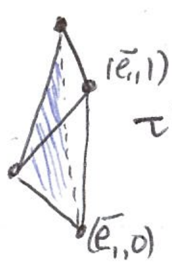
[$\forall a$, $\{v_i^{\,a}\}$ determines an $(n+1)$-simplex, $\tau^a : \Delta^{n+1} \longrightarrow \Delta^n \times [0,1]$]





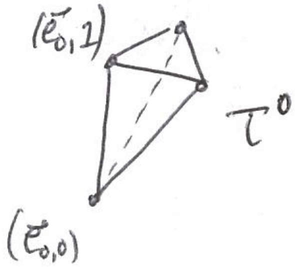
$(2,1)$

$(0,0) - (1,0) - (2,0)$



$(1,1) - (2,1)$

$(0,0) - (1,0)$



$(0,1) - (1,1) - (2,1)$

$(0,0)$

<u>Defn</u> $\forall\ \sigma \in \mathrm{Maps}(\Delta^n, X)$,

let

$$F_n\sigma = \sum (-1)^q \tau^q_\sigma,$$

where $\tau^q_\sigma : \Delta^{n+1} \longrightarrow Y$ is defined to be the map

$$\begin{tikzcd}
\Delta^{n+1} \arrow[r, "\tau^q"] \arrow[rrr, bend right] & \Delta^n \times [0,1] \arrow[r, "\sigma\times id"] & X\times[0,1] \arrow[r, "F"] & Y.
\end{tikzcd}$$

Need to show:

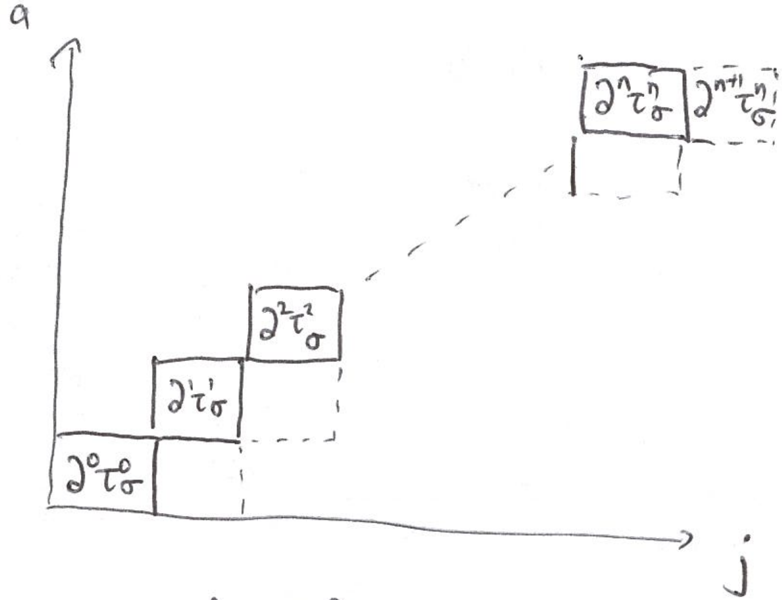
$$\partial F\sigma + F\partial\sigma = f_0(\sigma) - f_1(\sigma)$$

We write out the terms:

$$\partial F\sigma = \sum_{a,j} (-1)^j \partial^j \left((-1)^a \tau_\sigma^a \right)$$

$$= \sum_{a,j} (-1)^{a+j} \partial^j \tau_\sigma^a .$$

Graphically:



Note: sign of $(j,a)^{th}$ term is $(-1)^{a+j}$.

(plotting the summands of $\partial F\sigma$).

Exercise: $\partial^a \tau_\sigma^a = \partial^a \tau_\sigma^{a-1} \quad \forall\ 1 \le a \le n.$

Pf.

τ^a

$(a,1)$ - - - $(n,1)$

$(0,0)$ $(1,0)$ - - - $(a,0)$ a^{th} vertex

$\downarrow$ delete a^{th} vertex

τ^{a-1} a^{th} vertex

$(a-1,1)$ $(a,1)$ - - - $(n,1)$

$(0,0)$ - - - $(a-1,0)$

$\swarrow$ delete a^{th} vertex

$(a,1)$ - - - $(n,1)$

$(0,0)$ - - - $(a-1,0)$

Upshot:  terms cancel out.

How about $F\partial\sigma$?

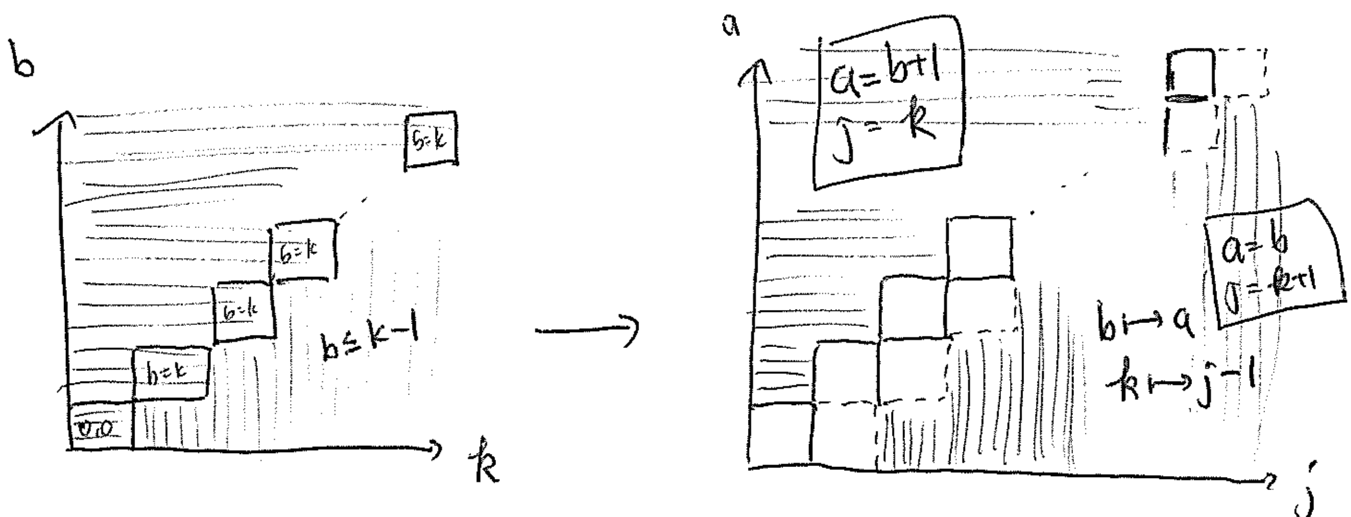
$$F\partial\sigma = \sum (-1)^b \tau^b_{\partial\sigma}$$

$$= \sum (-1)^{b+k} \tau^b_{\partial^k\sigma}.$$

<u>Exer:</u> $b \geq k \Rightarrow \tau^b_{\partial^k\sigma} = \partial^k \tau^{b+1}_{\sigma}$

$b \leq k-1 \Rightarrow \tau^b_{\partial^k\sigma} = \partial^{k+1}\tau^b_{\sigma}.$

So we can organize the (k,b) terms, align them w/ (j,a) terms:



And since $(-1)^{b+k}$ and $(-1)^{a+j} = (-1)^{b+k+1}$ differ by a sign, the shaded terms cancel.

The only terms that remain:

but $\partial^0 \tau^0_\sigma + (-1)^{n+n+1} \partial^{n+1} \tau^n_\sigma = f_1(\sigma) - f_0(\sigma)$.

So $$\partial F \sigma + F \partial \sigma = f_1(\sigma) - f_0(\sigma).$$

If you're bothered by sign, take $F = -H$, so you have

$$\partial H \sigma + H \partial \sigma = f_0(\sigma) - f_1(\sigma).$$

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