[Long exact sequences, relative homology, Excision, and Mayer-Vietoris].

Where do long exact sequences come from?

<u>Thm</u> Let $f: A \to B$, $g: B \to C$ be maps of chain complexes such that $\forall n$,

$$0 \longrightarrow A_n \xrightarrow{f} B_n \xrightarrow{g} C_n \longrightarrow 0$$

is a short exact sequence. Then $\exists$ a map $\partial_n : H_n(C) \to H_{n-1}(A)$, $\forall n$, such that

$$\cdots \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow \cdots$$

is a long exact sequence.

redundant adjective

(aka Snake Lemma).

You'll prove this in your homework.

Where would we possibly get such a pair of maps f and g?

Let $A \subset X$ be spaces. Note the inclusion

$$i : A \hookrightarrow X$$

induces a map of chain complexes

$$i : C_\bullet(A) \longrightarrow C_\bullet(X)$$

where $\forall n$,

$$C_n(A) \longrightarrow C_n(X)$$

is an injection.

<u>Defn</u>. $C_n(X, A) := C_n(X) / C_n(A)$.

<u>Lemma</u> The differential on $C_\bullet(X)$ induces a differential on $C_\bullet(X, A)$.

So we're precisely in the situation of the Theorem.

Let $H_n(X,A)$ be the n^{th} homology of $C_\bullet(X,A)$.

The theorem tells us $\exists$ a L.E.S.

$$\cdots \to H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \to H_{n-1}(A) \longrightarrow \cdots$$

Is there a topological meaning to $H_n(X,A)$?

<u>Yes</u>.

<u>Defn</u> $H_n(X,A)$ is called the <u>relative homology</u> of the pair (X,A)

Meaning 1: You'll prove this in your homework:

Thm (You) If $A \subset X$ is closed,

and $\exists$ some open set $U \subset X$

s.t. $A \subset U$ and U deformation

retracts onto A, then

$$H_n(X, A) \cong H_n(X/A) \quad \forall n.$$

homology of $C_n(X,A)$

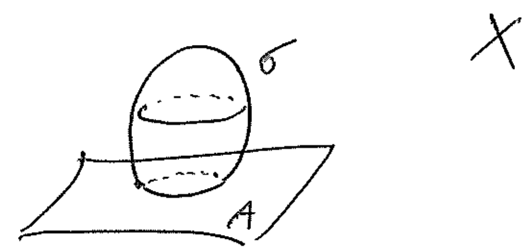
homology of space X/A.

The condition about U is technical-sounding, but is satisfied in most situations. For example, if $A_\bullet \subset X_\bullet$ is a sub-semisimplicial set, then $|A_\bullet| \subset |X_\bullet|$ satisfies this. It's not satisfied, for instance, if $A \subset X$ is a crazy dense subset (like $\mathbb{Q} \subset \mathbb{R}$).

⚠ $H_n(X/A) \neq H_n(X)/H_n(A)$.

<u>Meaning 2</u>. Let σ be a shape in X whose ∂ is in A.

∂ = boundary.



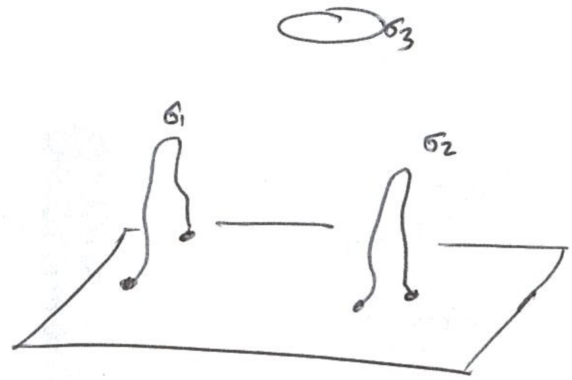
Then $\partial\sigma$ is in the image of $C.(A) \to C.(X)$, so $\partial\sigma$ defines an element of $H_n(X,A)$.

$\left[\text{So } H_n(X,A) \text{ captures geometry of shapes in } X \text{ whose boundary is contained in } A.\right]$

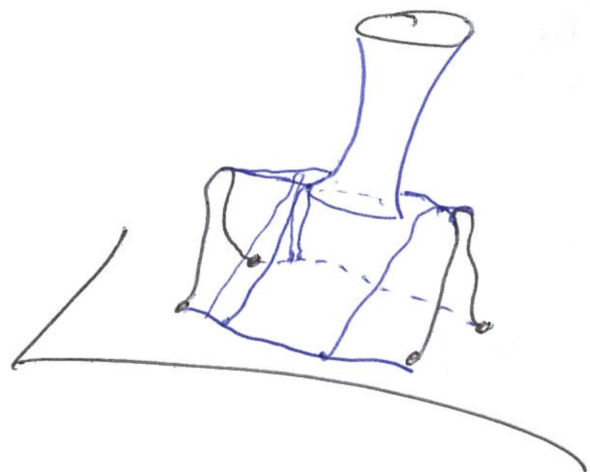
$$\begin{array}{ccc} \sigma & & \\ C_n(X) & \longrightarrow & C_n(X,A) \\ \downarrow \partial & & \downarrow \partial \\ C_{n-1}(X) & \longrightarrow & C_{n-1}(X,A). \\ \partial\sigma & & \end{array}$$

Ex $X = \mathbb{R}^3$

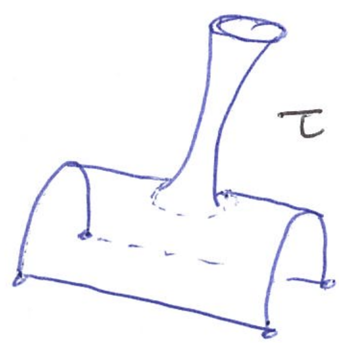
$A = \mathbb{R}^2$



Each of these curves defines a class in $H_1(X, A)$.



And σ_3 defines same homology class as $\sigma_1 + \sigma_2$.



Okay, but if $H_n(X,A)$ measures chains in X w/ boundary on A, will anything change if we thicken A at all? Or, if U is some thick subset, will taking away something on the inside of U change things?

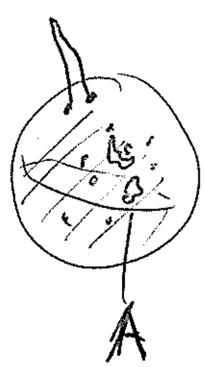
<u>Thm</u>. Let $A \subset U \subset X$ such that the closure of A is contained in the interior of U. Then the inclusion

$$X \setminus A \hookrightarrow X$$

induces an isomorphism in homology,

$$H_n(X \setminus A, U \setminus A) \cong H_n(X, U) \quad \forall n.$$

<u>Ex</u>: $X = \mathbb{R}^3$
U = ball
A = whatever inside ball, s.t. $\overline{A} \subset U$



Removing A won't change geometry of how chains in X can have ∂ in U.

Here's another application of the Snake Lemma.

Let $A, B \subset X$ w/ $\text{Interior}(A) \cup \text{Interior}(B) = X$.

There are two inclusions:

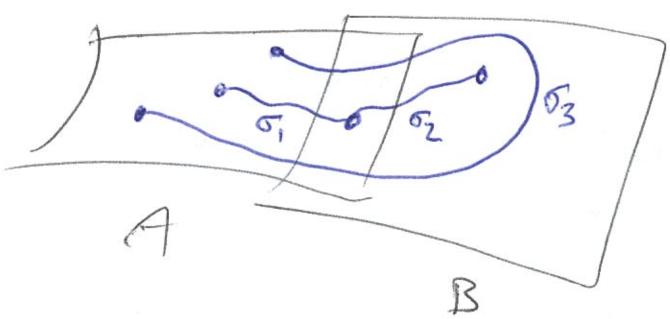
$$A \cap B \hookrightarrow A, \quad A \cap B \hookrightarrow B$$

which induce a map of chain complexes

$$C_\bullet(A \cap B) \longrightarrow C_\bullet(A) \oplus C_\bullet(B)$$

$\partial(a,b) = (\partial a, \partial b)$

Let $C_\bullet(A+B)$ be the sub-chain complex of $C_\bullet(X)$, where every element is a sum of simplices contained completely in A, or completely in B.



$\sigma_1 + \sigma_2 \in C_\bullet(A+B)$

$\sigma_3 \notin C_\bullet(A+B)$

<u>Prop.</u> $C_\bullet(A+B)$ is a chain complex,

and

$$0 \longrightarrow C_n(A\cap B) \longrightarrow C_n(A) \oplus C_n(B) \longrightarrow C_n(A+B) \longrightarrow 0$$
$$\sigma \longmapsto (\sigma, -\sigma)$$
$$(\alpha, \beta) \longmapsto \alpha+\beta$$

is a short exact sequence $\forall n$.

<u>Thm.</u> The inclusion

$$C_\bullet(A+B) \longrightarrow C_\bullet(X)$$

induces an $\cong$ on homology.

<u>Cor.</u> The Mayer-Vietoris sequence.

How does the connecting map

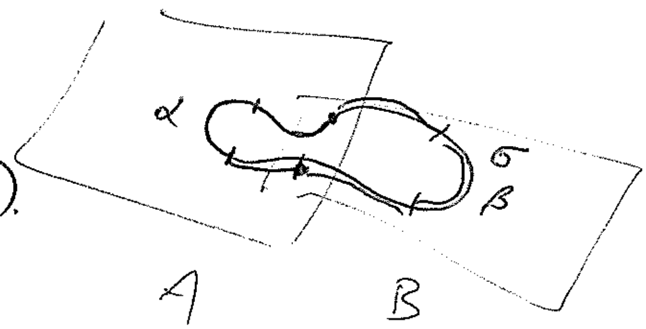
$$H_n(X) \longrightarrow H_{n-1}(A \cap B)$$

work? Given a cycle $\sigma \in \ker \partial_n^X$,

"replace it" by a cycle made of chains completely in A,

or completely in B, so

$$\sigma = \alpha + \beta + \partial^X(\text{something}).$$



Then $\partial^A \alpha$ (or $-\partial^B \beta$) is a cycle contained in $A \cap B$. //