Reduced homology

Formulas become clearer if we replace

$$H_0(X) \cong \mathbb{Z}^N$$

by $$\mathbb{Z}^{N-1}.$$

Defn. Consider the map

$$\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$$
$$\sum a_i x_i \longmapsto \sum a_i .$$

Then $$\operatorname{Ker}(\varepsilon) / \operatorname{image} \partial_1 =: \widetilde{H}_0(X),$$

and $$\widetilde{H}_n(X) := H_n(X).$$

The statement you're to prove for homework, for instance, is

Thm If $A \subset X$ is non-empty,

$$H_n(X, A) \cong \widetilde{H}_n(X/A) \quad \forall n.$$

From simplices to cells.

Semisimplicial sets are a very combinatorial way to build + describe spaces.
But they're not very flexible: For instance, would you be able to quickly write a semisimplicial set whose geometric realization is the n-sphere, S^n?

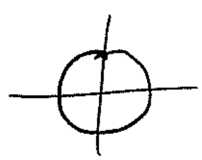
We'll now describe a more flexible way to describe and construct spaces, by gluing disks' boundaries to things.

Defn. The n-sphere is the space

$$S^n := \{ \vec{x} \in \mathbb{R}^{n+1} \mid |\vec{x}| = 1 \}.$$

Ex. $S^0 =$ 

, two points.

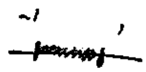
$S^1 =$  , circle

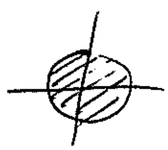
$S^2 =$  , sphere.

Defn The standard n-disk is the space

$$D^n := \{\vec{x} \in \mathbb{R}^n \mid |\vec{x}| \leq 1\}.$$

Ex. $D^0 = \mathbb{R}^0 = *$.

$D^1 =$  $=$ closed line interval.

$D^2 =$  $=$ closed disk $\subset \mathbb{R}^2$

$D^3 =$  $=$ closed unit ball $\subset \mathbb{R}^3$.

By ∂D^n, we mean the $(n-1)$-sphere. (I don't want to rigorously define "boundary" or "∂" yet.)

How might we build a space out of these?

(0) Start w/ a collection of <u>points</u>. i.e., a set called X^0.

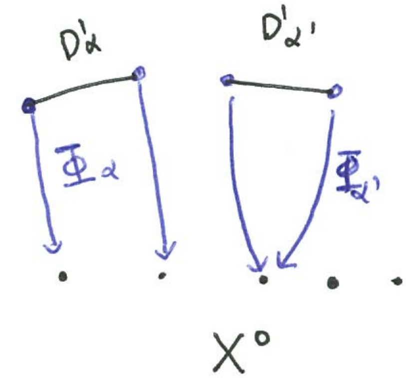
We call the elements of X^0 0-cells. (just as we called elements of X_0 0-<u>simplices</u>.)

(1) Choose a collection of 1-disks, $\{D^1_\alpha\}_{\alpha \in A_1^*}$ where A_1^* is just some indexing set. (It could be empty!)

And for each $\alpha \in A_1^*$, choose a continuous map

$$\Phi_\alpha : \partial D^1_\alpha \longrightarrow X^0.$$

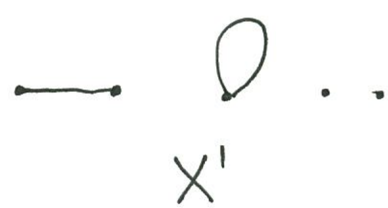
i.e., it's a collection of maps $S^0 \to X^0$.



Define

$$X^1 := \left(\coprod_{\alpha \in A_1} D^1_\alpha \right) \amalg X^0 \Big/ u \sim \Phi_\alpha(u) \;\; \forall\, u \in \partial D^1_\alpha.$$

A <u>space</u> not a set, so X^1 is a different animal from the "X_1" of a semisimplicial set.



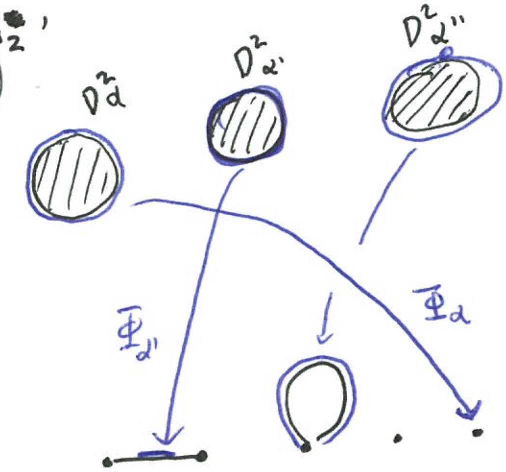
So far it looks just like a semisimplicial set, but things will change now.

(2) Choose a collection of 2-disks, $\{D^2_\alpha\}_{\alpha \in A_2}$, where A_2 is some indexing set. (Could be empty!)

$\forall\ \alpha \in A_2$, choose a map

$$\Phi_\alpha : \partial D^2_\alpha \longrightarrow X^1$$

i.e., a continuous map $S^1 \longrightarrow X^1$!



Define

$$X^2 := \left(\coprod_{\alpha \in A_2} D^2_\alpha\right) \amalg X^1 \Big/ \begin{array}{l} u \sim \Phi_\alpha(u) \\ \forall\ u \in \partial D^2_\alpha \end{array}.$$

You can continue this by induction:

Given a space X^{n-1}, choose a collection of (n)-disks, $\{D^n_\alpha\}_{\alpha \in A_n}$

some set; ~~NOT A to the n-th power~~ could be $\emptyset$.

$\forall \alpha \in A_n$, choose a map

$$\Phi_\alpha : \partial D^n_\alpha \longrightarrow X^{n-1}$$

and set

$$X^n := \left(\coprod_{\alpha \in A_n} D^n_\alpha \right) \amalg X^{n-1} \Big/ \begin{matrix} u \sim \Phi_\alpha(u) \\ \forall\, u \in \partial D^n_\alpha \end{matrix} .$$

Then define

$$X := \bigcup_{n \geq 0} X^n \qquad \left(\text{via the inclusions } \cdots \hookrightarrow X^{n-1} \hookrightarrow X^n \hookrightarrow \cdots\right).$$

We topologize X by saying U is open iff $U \cap X^n$ is open $\forall n$.

__Defn__ A space X obtained this way is called a __CW complex__.

__Rmk.__ Many times, the set A_n will be empty for all large n. For instance, any surface can be built w/ $A_3 = A_4 = \cdots = \emptyset$.

__Rmk__ Even if A_n is empty, A_{n+1} may not be!

$$\left(A_n = \emptyset \implies X^n \cong X^{n-1} \right).$$

__Defn__ The space $X^n \subset X$ is called the __n-skeleton__ of X. By an open n-cell, we mean the image of the interior of a disk D^n_{α} under the projection + inclusion

$$\left(\coprod D_\alpha^n\right) \amalg X^{n-1} \xrightarrow{h} X^n \hookrightarrow X.$$

By a closed n-cell, we mean D_α^n (or its image in X).

<u>Example</u> Every semisimplicial set $X_\bullet$ gives rise to a CW complex. How?

$\forall n$, let $X_{\leq n}$ be the semisimplicial set given by

$$(X_{\leq n})_i = X_i \quad \text{if } i \leq n$$
$$(X_{\leq n})_i = \phi \quad \text{if } i > n.$$

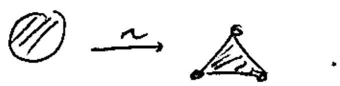
$$\phi \Lsh \cdots \phi \Rrightarrow X_n \Rrightarrow X_{n-1} \Rrightarrow \cdots \Rrightarrow X_0.$$

Then set

$$X^n = |X_{\leq n}|, \quad w/ \ A_n = X_n.$$

Concretely, choose a homeomorphism

$$D^n \cong \Delta^n$$



So we can write ∂D^n as a union $\bigcup_i \partial^i \Delta^n$.

The map $\Phi_\alpha : \partial D^n \longrightarrow X^{n-1}$

is given by $\Phi_\alpha\big|_{\partial^i \Delta^n}(u) = \Phi_\alpha \circ s_i^{\#}(v) = v.$

$$s_i : \Delta^{n-1} \hookrightarrow \partial\Delta^n.$$
$$u \longmapsto v$$

ie, though we draw ∂D^n as a round thing, we map it into X^{n-1} thinking of it as a piecewise linear shape.

<u>Rmk</u> If X is non-empty, then X^0 must be non-empty. Else $\nexists$ any maps $\partial D^1 \longrightarrow \phi \cong X^0$, so A_1 must be empty, and so on by induction. (To show X^0 empty $\Rightarrow A_n = \phi \; \forall n$.)

Less simplicial examples:

(1) Let $A_0 = \{*\}$, $A_n = \{\cdot\}$, $A_i = \phi$ otherwise.

So X will be a space w/ a single 0-cell and a single n-cell.

<u>$n=1$:</u> $\quad X^0 \amalg D^1_\alpha \big/ \sim \;\cong$  $\cong S^1$.

<u>$n=2$:</u> $\quad D^2_\alpha \amalg X^0 \big/ \bigcirc \sim \cdot \;\cong$  $\cong S^2$.

For general n, we see

$$X \cong D^n \amalg \cdot \big/ \sim \;\cong\; D^n \big/ \partial D^n \cong S^n.$$

Some more examples:

<u>Defn.</u>

$$\mathbb{R}P^n := \mathbb{R}^{n+1} \setminus \{0\} \big/ \mathbb{R}^\times$$

$$= \{\vec{x} \in \mathbb{R}^{n+1} \mid \vec{x} \neq 0\} \big/ {\vec{x} \sim \lambda \vec{x} \atop \forall \lambda \neq 0, \lambda \in \mathbb{R}}.$$

<u>Exer</u> Show

(1) $\mathbb{R}P^n \cong S^n \big/ \mathbb{Z}/2\mathbb{Z}$, $\quad \mathbb{Z}/2\mathbb{Z} \curvearrowright S^n$ by $x \mapsto -x$.

$$= \{\vec{x} \in S^n\} \big/ \vec{x} \sim -\vec{x}.$$

(2) $\mathbb{R}P^n$ is the space of all lines in $\mathbb{R}^{n+1}$ passing through the origin. i.e., the space of all dimension-one linear subspaces of $\mathbb{R}^{n+1}$.

Prop'n $\mathbb{R}P^n \cong D^n \amalg \mathbb{R}P^{n-1} \Big/ \vec{x} \sim [\vec{x}]$

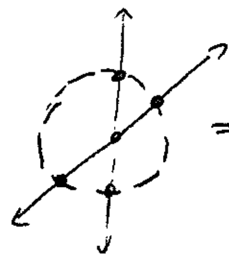
$\vec{x} \in \partial D^n,\ [\vec{x}] \in \mathbb{R}P^{n-1}$.

Ex

<u>n=0</u> $\mathbb{R}P^0$ = set of lines in $\mathbb{R}$

= pt.

<u>n=1</u> $\mathbb{R}P^1$ = ?

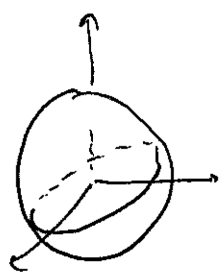


$\Longrightarrow$ $\mathbb{R}P^1 \cong$  upper hemisphere / two endpoints

since each endpoint defines the same line in $\mathbb{R}^2$

$\cong D^1 \big/ \partial D^1 \cong S^1$.

<u>n=2</u>



S^3 modulo $\vec{x} \sim -\vec{x}$

S||

upper hemisphere / $\vec{x} \sim -\vec{x}$ on equator

S||

$D^2 \Big/ \vec{x} \sim -\vec{x}$ on ∂D^2

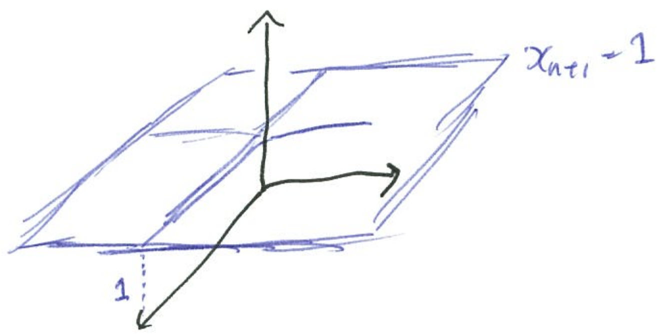
In general, we always have

$$\text{upper hemisphere} \cong D^n \Big/ \begin{array}{l}\vec{x} \sim -\vec{x} \\ \text{on } \partial D^n.\end{array}$$

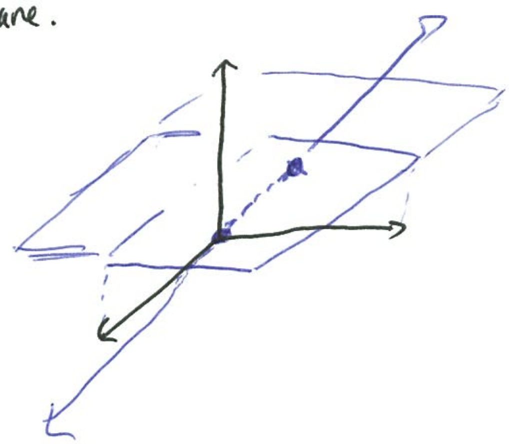
//

<u>Rmk.</u> The drawing I always keep in mind:

In $\mathbb{R}^{n+1}$, fix the (affine) plane $x_{n+1} = 1$.



$\forall$ lines $L \subset \mathbb{R}^{n+1}$, examine the intersection of L w/ this plane.



If the direction vector of L has a non-zero x_{n+1}-component, L defines a unique pt of the plane. Otherwise, L is a point of $\mathbb{RP}^{n-1}$. So we see that as a set,

$$\mathbb{RP}^n = \mathbb{R}^n \amalg \mathbb{RP}^{n-1}$$

identified w/ our plane; interior of D^n.

__Defn.__

$$\mathbb{CP}^n := \mathbb{C}^{n+1}\setminus\{0\} \big/ \mathbb{C}^\times$$

$$= \{(z_0,\cdots,z_n) \mid \text{some } z_i \text{ is non-zero}\}$$

modulo $(z_0,\cdots,z_n) \sim (\lambda z_0,\cdots,\lambda z_n) \quad \forall \lambda\in\mathbb{C}\setminus\{0\}$.

__Exer.__ Show

(1) $\mathbb{CP}^n \cong S^{2n+1}/S^1$

$$= \{(z_0,\cdots,z_n) \mid \Sigma \|z_i\|^2 = 1\} \big/ \vec{z}\sim e^{i\theta}\vec{z}$$

(2) $\mathbb{CP}^n$ = the space of complex lines through the origin of $\mathbb{C}^{n+1}$.

As w/ $\mathbb{RP}^n$ case, we can give a CW structure on $\mathbb{CP}^n$ using $\mathbb{CP}^{n-1}$.

Prop $\mathbb{C}P^n \cong D^{2n} \amalg \mathbb{C}P^{n-1} \Big/ \begin{array}{l} \vec{z} \sim [\vec{z}] \\ \forall\, \vec{z} \in \partial D^{2n} = S^{2n-1}, \\ [\vec{z}] \in \mathbb{C}P^{n-1}. \end{array}$

Pf. Consider $D^{2n} = \{(z_0, \dots, z_{n-1}) \mid \sum |z_i|^2 \leq 1\}$

and the map

$$p: D^{2n} \longrightarrow \mathbb{C}P^n$$
$$(z_0, \dots, z_{n-1}) \longmapsto \Big[\Big(z_0, \dots, z_{n-1}, \sqrt{1 - \sum_{0 \leq i \leq n-1} |z_i|^2}\Big)\Big].$$

This is a surjection: Thinking of $\mathbb{C}P^n \cong S^{2n+1}/S^1$, we see that every $\vec{z} \in S^{2n+1}$ is of the form

$$\Big(\lambda z_0, \dots, \lambda z_{n-1}, \lambda \sqrt{1 - \sum |z_i|^2}\Big) \text{ for some } (z_0, \dots, z_{n-1}) \in D^{2n}, \ \lambda \in S^1.$$

It's an injection everywhere except when $\sum |z_i|^2 = 1$; i.e., when $(z_0, \dots, z_{n-1}) \in \partial D^{2n} \cong S^{2n-1}$. Then,

$$p(z_0, \dots, z_{n-1}) = p(z_0', \dots, z_{n-1}')$$

iff $z_i = e^{i\theta} z_i' \ \forall i$, some fixed θ. This is precisely the identification that maps ∂D^{2n} to $\mathbb{C}P^{n-1}$! //

Lemma. If X is a CW complex,

(a) $$H_k(X^n, X^{n-1}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z}^{A_n} & k = n \end{cases}$$

A_n: set of n-cells of X.

Why 0 when $k=0$? $H_0(X,A)$ should be reduced homology of X/A, so should be zero if X/A is connected.

(b) $H_k(X^n) = 0 \quad \forall\, k > n.$

No higher-dimensional holes/shapes in something low-dimensional.

(c) $\forall\, k<n$, the inclusion

$$i : X^n \longrightarrow X$$

induces an $\cong$ on H_k.

n-skeleton contains all lower homology of X.

The $(n+1)$-skeleton can still alter H_n, since it can kill some homology by making things boundaries.

Pf. (a) Note that $A = X^{n-1}$ is a def. retract of an open set in X^n. $\forall \alpha \in A_n$, consider some ε-neighborhood $U_\alpha \supset \partial D^n_\alpha$.

D^2_α

U_α clearly retracts onto ∂D^2_α. Taking

$$U = A \cup \left(\bigcup_{\alpha \in A_n} U_\alpha \right) \subset X^n$$

the hypotheses of your homework hold.

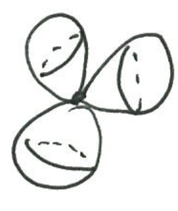
Hence $H_k(X^n, X^{n-1}) \cong \widetilde{H}_k(X_n / X_{n-1})$.

$$\text{But } X_n / X_{n-1} \cong \left. \left(\coprod_\alpha D_\alpha^n\right) \amalg X^{n-1} / \sim \right/ X^{n-1}$$

$$\cong \coprod_\alpha D_\alpha^n \Big/ \begin{matrix} u_\alpha \sim u_\beta \\ u_\alpha \sim u_\alpha' \end{matrix} \quad \forall\, u_\alpha \in \partial D_\alpha$$

$$\cong \bigvee_\alpha S^n_\alpha .$$

wedge sum



By Mayer-Vietoris,

$$\widetilde{H}_\bullet\left(\bigvee_\alpha S^n_\alpha\right) \cong \bigoplus_\alpha \widetilde{H}_\bullet(S^n) \cong \begin{cases} 0 & \bullet \neq n \\ \mathbb{Z}^{A_n} & \bullet = n. \end{cases}$$

(b) By LES of a pair,

$$\begin{tikzcd} & & H_{k+1}(X^n, X^{n-1}) \arrow[dll, rounded corners] \\ H_k(X^{n-1}) \arrow[r] & H_k(X^n) \arrow[r] & H_k(X^n, X^{n-1}) \arrow[dll, rounded corners] \\ {} & & \end{tikzcd}$$

is exact. But $k > n \Rightarrow H_k(X^n, X^{n-1}) \cong \tilde{H}_k(S^n \vee \cdots \vee S^n) \cong 0.$

So $H_k(X^{n-1}) \to H_k(X^n)$ is an isomorphism.

$k > n \Rightarrow k > n-1$, so we have

$$H_k(X^0) \cong H_k(X^1) \cong \cdots \cong H_k(X^{n-1}) \cong H_k(X^n)$$

but $H_k(X^0) \cong H_k(\text{bunch of points}) \cong 0$ for $k > 0$.

(c) First prove that for $m > n$,

$$X^n \longrightarrow X^m$$

induces an $\cong$ on all H_k for $k < n$.

By the same LES

$$\begin{tikzcd} & & H_{k+1}(X^n, X^{n-1}) \arrow[dll, bend left] \\ H_k(X^{n-1}) \arrow[r] & H_k(X^n) \arrow[r] & H_k(X^n, X^{n-1}) \arrow[dll, bend left] \\ {} & & \end{tikzcd}$$

we know $k \neq n \Rightarrow H_k(X^n, X^{n-1}) \cong \tilde{H}_k(\vee S^n) \cong 0$.

Hence

$$k \neq n \to H_k(X^{n-1}) \cong H_k(X^n).$$

But $k < n \Rightarrow k < n+1$, so

$$(*) \qquad H_k(X^{n-1}) \cong H_k(X^n) \cong H_k(X^{n+1}) \cong \cdots \cong H_k(X^m)$$

$\forall\, m > n$.

Now, if $X \neq X^m$ for some m finite (i.e., if X is ∞-dim), let $\sum a_i \sigma_i$ be a singular k-~~chain~~ cycle in X. Each σ_i is a map $\sigma_i : \Delta^k \to X$, but Δ^k is compact, so it intersects only finitely many cells of X. i.e., $\sigma_i(\Delta^k) \subset X^{m_i}$ for some m_i. Then let $\max_i m_i = m$; by $(*)$, $\sum a_i \sigma_i$ is homologous to something in X^n?

So $i_* : H_k(X^n) \to X$ is a surjection for $k<n$.

It's an injection because if

$$\partial\left(\sum b_i \tau_i\right) = \sum a_i \sigma_i,$$

again $\bigcup_i \tau_i(\Delta^n) \subset X^m$ for some m finite. But $H_k(X^n) \cong H_k(X^m)$

$\forall\ k < n \leq m$. //

Now we can define <u>cellular homology</u>, or <u>CW homology</u>.

We ~~…~~ use the degree n^{th} part of LES of pairs for (X^n, X^{n-1}), $\forall n$.

$$\cdots \to \overset{\overset{0}{\cong}\ (a)}{H_{n+2}(X^{n+1},X^n)} \to \overset{\overset{0}{\cong}\ \text{by (b)}}{H_{n+1}(X^n)} \to H_{n+1}(X^{n+1}) \to H_{n+1}(X^{n+1},X^n) \xrightarrow{\partial_{n+1}} H_n(X^n) \to \overset{\overset{H_n(X)}{\cong}\ (c)}{H_n(X^{n+1})} \to \overset{\overset{0}{\cong}\ (a)}{H_n(X^{n+1},X^n)} \to \cdots$$

$$\cdots \to \overset{\overset{0}{\cong}\ (a)}{H_{n+1}(X^n,X^{n-1})} \to \overset{\overset{0}{\cong}\ \text{by (b)}}{H_n(X^{n-1})} \to H_n(X^n) \xrightarrow{j_n} H_n(X^n,X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \to \overset{\overset{H_{n-1}(X)}{\cong}\ (c)}{H_{n-1}(X^n)} \to \overset{\overset{0}{\cong}\ (a)}{H_{n-1}(X^n,X^{n-1})} \to \cdots$$

$$\overset{\overset{0}{\cong}\ (a)}{H_n(X^{n-1},X^{n-2})} \to \overset{\overset{0}{\cong}\ \text{by (b)}}{H_{n-1}(X^{n-2})} \to H_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} H_{n-1}(X^{n-1},X^{n-2}) \xrightarrow{\partial_{n-1}} H_{n-2}(X^{n-2}) \to \overset{\overset{H_{n-2}(X)}{\cong}\ (c)}{H_{n-2}(X^{n-1})} \to \overset{\overset{0}{\cong}\ (a)}{H_{n-2}(X^{n-1},X^{n-2})} \to \cdots$$

$=$

$=$

<u>**Defn**</u> $H_{n+1}(X^{n+1},X^n) \xrightarrow{d_{n+1}} H_n(X^n,X^{n-1})$, $d_{n+1} := j_n \circ \partial_{n+1}$

defines a differential. By Lemma (a),

$$\begin{matrix} \vdots \\ \downarrow \\ \mathbb{Z}^{A_{n+1}} \\ \downarrow d_{n+1} \\ \mathbb{Z}^{A_n} \\ \downarrow d_n \\ \mathbb{Z}^{A_{n-1}} \\ \downarrow d_{n-1} \\ \vdots \end{matrix}$$

cellular chain complex of X.

Note $d \circ d = 0$ since

$$d_n \circ d_{n+1} = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1}$$

$$= j_{n-1} \circ 0 \circ \partial_{n+1}$$

since ∂_n, j_n are successive maps in a LES

So $d_{n+1} : H_{n+1}(X^{n+1}, X^n) \longrightarrow H_n(X^n, X^{n-1})$ defines a differential for a chain complex. By Lemma (a),

$$H_n(X^n, X^{n-1}) \cong \mathbb{Z}^{A_n},$$ so

we have a chain complex

$$\begin{array}{c} \vdots \\ \big\downarrow d_{n+1} \\ \mathbb{Z}^{A_n} \\ \big\downarrow d_n \\ \mathbb{Z}^{A_{n-1}} \\ \big\downarrow d_{n-1} \\ \vdots \end{array}$$

called the <u>cellular chain complex</u>.

Def'n. $H_n^{CW}(X) := H_n\left(\begin{array}{c} \vdots \\ \downarrow \\ \mathbb{Z}^{A_n} \\ \downarrow \partial_n \\ \mathbb{Z}^{A_{n-1}} \\ \downarrow \\ \vdots \end{array} \right)$,

is called the <u>cellular</u> <u>homology</u> of the CW complex X.

Prop'n

$$H_n^{CW}(X) \cong H_n(X) \quad \forall\, n \geq 0.$$

Pf. By exactness of

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}} H_n(X^n) \longrightarrow \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ H_n(X) \text{ by (c)}}}{H_n(X^{n+1})} \longrightarrow 0$$

we see $H_n(X) \cong H_n(X^n) / \operatorname{image}(\partial_{n+1})$.

By exactness of

$$\overset{\substack{0 \text{ by (b)} \\ \cong}}{H_n(X^{n-1})} \longrightarrow H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1})$$

j_n is injective, so

- $\operatorname{image}(d_{n+1}) \cong \operatorname{image}(j_n \circ \partial_{n+1}) \cong \operatorname{image}(\partial_{n+1})$, and
- $\operatorname{image}(j_n) \cong H_n(X^n)$.

But $\text{image}(j_n) \cong \text{Ker}(\partial_n)$ by exactness

And $\text{Ker}\, d_n = \text{Ker}(j_{n-1} \circ \partial_n) = \text{Ker}(\partial_n)$ since j_{n-1} is injective. //

$\boxed{\text{Summary:}}$

$$H_n^{CW}(X) \cong \text{Ker}\, d_n / \text{Image}\, d_{n+1}.$$

$$\begin{aligned}
\text{Ker}\, d_n &\cong \text{Ker}(j_{n-1} \circ \partial_n) \\
&\cong \text{Ker}(\partial_n) && j \text{ injective} \\
&\cong \text{image}(j_n) && \text{exactness} \\
&\cong H_n(X^n) && j_n \text{ injective}
\end{aligned}$$

$$\begin{aligned}
\text{Image}(d_{n+1}) &\cong \text{Image}(j_n \circ \partial_{n+1}) \\
&\cong \text{Image}(\partial_{n+1}) && j \text{ injective.}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad H_n^{CW}(X) &\cong H_n(X^n) / \text{image}(\partial_{n+1}) \\
&\cong H_n(X^{n+1}) && \text{by exactness} \\
&\cong H_n(X) && \text{by } (\bullet). \quad //
\end{aligned}$$