Lemma. If X is a CW complex,

(a) $$H_k(X^n, X^{n-1}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z}^{A_n} & k = n \end{cases}$$

A_n: set of n-cells of X.

Why 0 when $k=0$? $H_0(X,A)$ should be reduced homology of X/A, so should be zero if X/A is connected.

(b) $H_k(X^n) = 0 \quad \forall\, k > n.$

No higher-dimensional holes/shapes in something low-dimensional.

(c) $\forall\, k<n$, the inclusion

$$i : X^n \longrightarrow X$$

induces an $\cong$ on H_k.

n-skeleton contains all lower homology of X.

The $(n+1)$-skeleton can still alter H_n, since it can kill some homology by making things boundaries.

Pf. (a) Note that $A = X^{n-1}$ is a def. retract of an open set in X^n. $\forall \alpha \in A_n$, consider some ε-neighborhood $U_\alpha \supset \partial D^n_\alpha$.

D^2_α

U_α clearly retracts onto ∂D^2_α. Taking

$$U = A \cup \left(\bigcup_{\alpha \in A_n} U_\alpha \right) \subset X^n$$

the hypotheses of your homework hold.

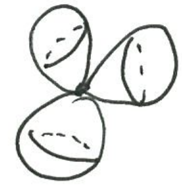
Hence $H_k(X^n, X^{n-1}) \cong \widetilde{H}_k\left(X_n / X_{n-1} \right)$.

$$\text{But } X_n / X_{n-1} \cong \left(\coprod_\alpha D^n_\alpha \right) \amalg X^{n-1} /_\sim \Big/ X^{n-1}$$

$$\cong \coprod_\alpha D^n_\alpha \Big/ \begin{matrix} u_\alpha \sim u_\beta \quad \forall u_\alpha \in \partial D_\alpha \\ u_\alpha \sim u'_\alpha \end{matrix}$$

$$\cong \bigvee_\alpha S^n_\alpha .$$

wedge sum



By Mayer-Vietoris,

$$\widetilde{H}_\bullet\left(\bigvee_\alpha S^n_\alpha \right) \cong \bigoplus_\alpha \widetilde{H}_\bullet(S^n) \cong \begin{cases} 0 & \bullet \neq n \\ \mathbb{Z}^{A_n} & \bullet = n. \end{cases}$$

(b) By LES of a pair,

$$\cdots \to H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \longrightarrow H_k(X^n) \to H_k(X^n, X^{n-1}) \to \cdots$$

is exact. But $k > n \Rightarrow H_k(X^n, X^{n-1}) \cong \tilde{H}_k(S^n \vee \cdots \vee S^n) \cong 0$.

So

$$H_k(X^{n-1}) \to H_k(X^n) \text{ is an isomorphism.}$$

$k > n \Rightarrow k > n-1$, so we have

$$H_k(X^0) \cong H_k(X^1) \cong \cdots \cong H_k(X^{n-1}) \cong H_k(X^n)$$

but $H_k(X^0) \cong H_k(\text{bunch of points}) \cong 0$ for $k > 0$.

(c) First prove that for $m > n$,

$$X^n \longrightarrow X^m$$

induces an $\cong$ on all H_k for $k < n$.

By the same LES

$$\begin{tikzcd} & & H_{k+1}(X^n, X^{n-1}) \arrow[dll] \\ H_k(X^{n-1}) \arrow[r] & H_k(X^n) \arrow[r] & H_k(X^n, X^{n-1}) \arrow[dll] \\ {} & & \end{tikzcd}$$

we know $k \neq n \Rightarrow H_k(X^n, X^{n-1}) \cong \tilde{H}_k(\vee S^n) \cong 0$.

Hence

$$k \neq n \rightarrow H_k(X^{n-1}) \cong H_k(X^n).$$

But $k < n \Rightarrow k < n+1$, so

$$(*) \qquad H_k(X^{n-1}) \cong H_k(X^n) \cong H_k(X^{n+1}) \cong \cdots \cong H_k(X^m)$$

$\forall \; m > n$.

Now, if $X \neq X^m$ for some m finite (i.e., if X is ∞-dim), let $\sum a_i \sigma_i$ be a singular k-~~chain~~ cycle in X. Each σ_i is a map $\sigma_i : \Delta^k \longrightarrow X$, but Δ^k is compact, so it intersects only finitely many cells of X. i.e., $\sigma_i(\Delta^k) \subset X^{m_i}$ for some m_i. Then let $\max_i m_i = m$; by $(*)$, $\sum a_i \sigma_i$ is homologous to something in X^n?

So $i_* : H_k(X^n) \longrightarrow X$ is a surjection for $k<n$.

Its an injection because if

$$\partial\left(\sum b_i \tau_i\right) = \sum a_i \sigma_i,$$

again $\bigcup_i \tau_i(\Delta^n) \subset X^m$ for some m finite. But $H_k(X^n) \cong H_k(X^m)$

$\forall\ k<n\leq m.$ //

Now we can define <u>cellular homology</u>, or <u>CW homology</u>.

We use the degree n^{th} bit of LES of pairs for (X^n, X^{n-1}), $\forall n$.

$$\cdots \to \overset{(a)\ \overset{0}{\cong}}{H_{n+2}(X^{n+1},X^n)} \to \overset{\overset{0}{\cong}\text{ by (b)}}{H_{n+1}(X^n)} \to H_{n+1}(X^{n+1}) \to H_{n+1}(X^{n+1},X^n) \xrightarrow{\partial_{n+1}} H_n(X^n) \to \overset{\overset{H_n(X)}{\cong}\ (c)}{H_n(X^{n+1})} \to \overset{\overset{0}{\cong}\ (a)}{H_n(X^{n+1},X^n)} \to \cdots$$

$$\cdots \to \overset{(a)\ \overset{0}{\cong}}{H_{n+1}(X^n,X^{n-1})} \to \overset{\overset{0}{\cong}\text{ by (b)}}{H_n(X^{n-1})} \to H_n(X^n) \xrightarrow{j_n} H_n(X^n,X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \to \overset{\overset{H_{n-1}(X)}{\cong}\ (c)}{H_{n-1}(X^n)} \to \overset{\overset{0}{\cong}\ (a)}{H_{n-1}(X^n,X^{n-1})} \to \cdots$$

$$\overset{(a)\ \overset{0}{\cong}}{H_n(X^{n-1},X^{n-2})} \to \overset{\overset{0}{\cong}\text{ by (b)}}{H_{n-1}(X^{n-2})} \to H_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} H_{n-1}(X^{n-1},X^{n-2}) \xrightarrow{\partial_{n-1}} H_{n-2}(X^{n-2}) \to \overset{\overset{H_{n-2}(X)}{\cong}\ (c)}{H_{n-2}(X^{n-1})} \to \overset{\overset{0}{\cong}\ (a)}{H_{n-2}(X^{n-1},X^{n-2})} \to \cdots$$

Unnecessary!

Defn $\quad H_{n+1}(X^{n+1},X^n) \xrightarrow{d_{n+1}} H_n(X^n,X^{n-1})$, $\quad d_{n+1} := j_n \circ \partial_{n+1}$

defines a differential. By Lemma (a),

$$\begin{array}{c} \vdots \\ \downarrow \\ \mathbb{Z}^{A_{n+1}} \\ \downarrow d_{n+1} \\ \mathbb{Z}^{A_n} \\ \downarrow d_n \\ \mathbb{Z}^{A_{n-1}} \\ \downarrow d_{n-1} \\ \vdots \end{array}$$

cellular chain complex of X.

Note $d \circ d = 0$ since

$$d_n \circ d_{n+1} = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1}$$

$$= j_{n-1} \circ 0 \circ \partial_{n+1}$$

since ∂_n, j_n are successive maps in a LES.

So $d_{n+1} : H_{n+1}(X^{n+1}, X^n) \longrightarrow H_n(X^n, X^{n-1})$ defines a differential for a chain complex. By Lemma (a),

$$H_n(X^n, X^{n-1}) \cong \mathbb{Z}^{A_n},$$ so

we have a chain complex

$$\begin{array}{c} \vdots \\ \downarrow d_{n+1} \\ \mathbb{Z}^{A_n} \\ \downarrow d_n \\ \mathbb{Z}^{A_{n-1}} \\ \downarrow d_{n-1} \\ \vdots \end{array}$$

called the <u>cellular</u> <u>chain complex</u>.

Defn. $H_n^{CW}(X) := H_n\left(\begin{array}{c}\vdots\\ \downarrow\\ \mathbb{Z}^{A_n}\\ \downarrow \partial_n\\ \mathbb{Z}^{A_{n-1}}\\ \downarrow\\ \vdots\end{array}\right)$,

is called the <u>cellular homology</u> of the CW complex X.

Propn

$$H_n^{CW}(X) \cong H_n(X) \quad \forall n \geq 0.$$

Pf. By exactness of

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}} H_n(X^n) \to \overset{\substack{H_n(X) \text{ by (c)} \\ \wr\wr}}{H_n(X^{n+1})} \to 0$$

we see $H_n(X) \cong H_n(X^n) / \text{image}(\partial_{n+1})$.

By exactness of

$$\overset{\substack{0 \text{ by (b)} \\ \wr\wr}}{H_n(X^{n-1})} \to H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1})$$

j_n is injective, so

- $\text{image}(d_{n+1}) \cong \text{image}(j_n \circ \partial_{n+1}) \cong \text{image}(\partial_{n+1})$, and
- $\text{image}(j_n) \cong H_n(X^n)$.

But $\text{image}(j_n) \cong \text{Ker}(\partial_n)$ by exactness

And $\text{Ker}\, d_n = \text{Ker}(j_{n-1} \circ \partial_n) = \text{Ker}(\partial_n)$ since j_{n-1} is injective. //

$\boxed{\text{Summary!}}$

$$H_n^{CW}(X) \cong \text{Ker}\, d_n / \text{Image}\, d_{n+1}.$$

$$\text{Ker}\, d_n \cong \text{Ker}(j_{n-1} \circ \partial_n)$$

$$\cong \text{Ker}(\partial_n) \qquad j \text{ injective}$$

$$\cong \text{image}(j_n) \qquad \text{exactness}$$

$$\cong H_n(X^n) \qquad j_n \text{ injective}$$

$$\text{Image}(d_{n+1}) \cong \text{Image}(j_n \circ \partial_{n+1})$$

$$\cong \text{Image}(\partial_{n+1}) \qquad j \text{ injective.}$$

$$\Rightarrow \quad H_n^{CW}(X) \cong H_n(X^n) / \text{image}(\partial_{n+1})$$

$$\cong H_n(X^{n+1}) \qquad \text{by exactness}$$

$$\cong H_n(X) \qquad \text{by } (*). \qquad //$$

$$H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}).$$

Where does this map come from?

$$\begin{array}{ccccc} & & B_n & \xrightarrow{g} & C_n \\ & & \downarrow{\scriptstyle \partial^B} & & \downarrow{\scriptstyle \partial^C} \\ A_{n-1} & \xrightarrow{f} & B_{n-1} & \xrightarrow{g} & C_{n-1} \end{array}$$

If $\partial^C c = 0$, choose b s.t. $g(b) = c$. Then $\partial^C g(b) = g(\partial^B b)$ so $\partial^B b \in \operatorname{Ker} g_{n-1}$, hence $\exists$ a s.t. $f(a) = \partial^B b$. This is the map $H_\cdot(C) \to H_\cdot(A)$.

In geometric terms, if $b \in C_n(X^n)$ has boundy in X^{n-1}, then $[b] \in H_\cdot(X^n, X^{n-1})$ is sent to $[\partial^{X^n} b] \in H_{n-1}(X^{n-1})$.

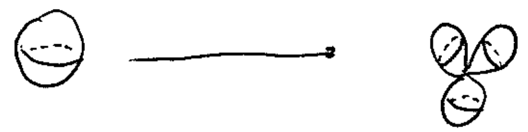
i.e., this is the map that sends an n-cell D^n_α (a generator of $H_n(X^n, X^{n-1})$) to the image of its boundy, $\Phi_\alpha(\partial D^n_\alpha) \subset X^{n-1}$.

$$\begin{array}{ccc} & & H_{n-1}(X^{n-1}) \\ & \swarrow{\scriptstyle =} & \\ H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2}) & & \end{array}$$

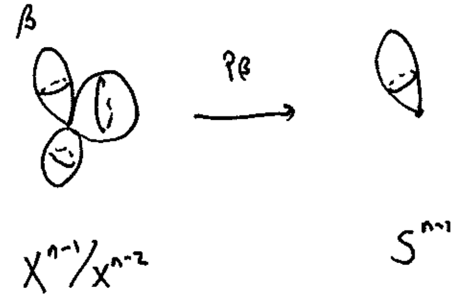
And this map? It sends a cycle in X^{n-1} to a cycle in the wedge of spheres, $X^{n-1}/X^{n-2} \cong \bigvee_{\beta \in A_{n-1}} S^{n-1}_\beta$.

All told, the geometric interpretation is: What is the image (in homology) of the composite map

$$f_\alpha : \partial D^n_\alpha \longrightarrow X^{n-1} \longrightarrow X^{n-1}/X^{n-2}$$



Going further, collapse everything in X^{n-1}/X^{n-2} to a point, except for some cell interior(D^{n-1}_β):



Define $f_{\alpha\beta}: S^{n-1} \cong \partial D^n_\alpha \longrightarrow S^{n-1}$ by composite $p_\beta \circ f_\alpha$. Since these p_β realize the direct sum

$$\tilde{H}_\bullet(X^{n-1}/X^{n-2}) \cong \bigoplus_{\beta \in A_{n-1}} \tilde{H}_\bullet(S^{n-1})$$

we now have a formula for d_n:

<u>Prop'n</u> (Cellular boundary formula)

The map

$$\begin{array}{ccc} d_n: H_n(X^n, X^{n-1}) & \longrightarrow & H_{n-1}(X^{n-1}, X^{n-2}) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}A_n & & \mathbb{Z}A_{n-1} \\ \| & & \\ \left\{ \begin{array}{l} \sum n_i \alpha_i \text{ s.t.} \\ n_i \in \mathbb{Z},\ \alpha_i \in A_n, \\ \text{only finitely} \\ \text{many } n_i \text{ are} \\ \text{non-zero} \end{array} \right\} & & \end{array}$$

is given by

$$d_n(\alpha) = \sum_{\beta \in A_{n-1}} \deg(f_{\alpha\beta})\beta.$$

What do I mean by degree?

$$f_{\alpha\beta} : S^{n-1} \longrightarrow S^{n-1}$$

is a map from a sphere to a sphere. Well, on top homology, $f_{\alpha\beta}$ induces a map

$$(f_{\alpha\beta})_* : H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(S^{n-1}).$$

Since $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$, such a linear map is just "multiplication by d" for some $d \in \mathbb{Z}$.

$$(f_{\alpha\beta})_* \text{ "is" } \mathbb{Z} \xrightarrow{\times d} \mathbb{Z}.$$

d is called the <u>degree</u> of $f_{\alpha\beta}$.

<u>Rmk</u>. We've <u>chosen</u> a homeomorphism $\widetilde{H}_\bullet(D^{n-1}, \partial D^{n-1}) \xrightarrow{\sim} \widetilde{H}_\bullet(S^{n-1})$ throughout this discussion. These choices are important, since an orientation reversal can <u>change</u> the sign of the degree of a map.