Homotopy Groups

What have we done so far?
Defined a sequence of functors

$$H_n : \text{Spaces} \to \text{Abelian Groups}$$
$$X \longmapsto H_n(X)$$

called n^{th} homology.

We'll discuss more formal properties in a second.

What we begin today is the definition and study of another invariant, the <u>homotopy groups</u>.

These will be a sequence of functors from the category of <u>pointed</u> spaces to the category of groups,

$$\pi_n : \text{Spaces}_* \to \text{Groups}$$

for all $n \geq 1$.

Def'n A pointed space is a space X together w/ a choice of basepoint $x_0 \in X$.
A <u>map</u> of pointed spaces

$$f : (X, x_0) \to (Y, y_0)$$

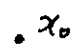
is a continuous map $f : X \to Y$ such that

$$f(x_0) = y_0 .$$

Ex Here are some pointed spaces:

$X = S^2$

$X = S^2 \amalg D^2$ (X can be disconnected)



$X = pt$

Note $X = \emptyset$ is NOT a pointed set.

Def'n Spaces_* is the category of pointed spaces: objects are pointed spaces (X, x_0) and morphisms are maps of pointed spaces.
Composition is composition of continuous maps.

Exer Let $Y = * = y_0$.

Show that

$$\hom(Y, (X, x_0))$$

and

$$\hom((X, x_0), Y)$$

are one-point sets.

Rmk So $*$ is like the object $\mathbb{1}$ in the category of groups, or the object 0 in the category of chain complexes. There's only one map to and from this object.

Defn A homotopy between two maps $f_0, f_1 : (X, x_0) \to (Y, y_0)$ is a continuous map

$$F : X \times I \longrightarrow Y$$

such that

$$F(x_0, t) = y_0 \ \forall t,$$

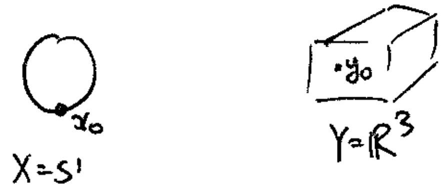
$$F(-, 0) = f_0$$

$$F(-, 1) = f_1.$$

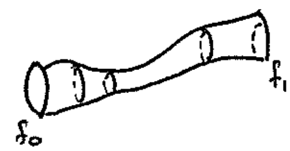
If such an F exists, we say f_0 and f_1 are homotopic (as pointed maps).

↑ often omitted.

Ex



A homotopy of unpointed maps:



A homotopy of pointed maps:

x_0 is always constrained.

More generally,

Defn Let $A \subset X$. A homotopy relative to A, or homotopy rel A, between maps $f_0, f_1 : X \longrightarrow Y$ is a homotopy $F : X \times I \longrightarrow Y$ such that

$$F(a, t) = F(a, 0) \quad \forall\, a \in A,\ t \in [0,1].$$

Exer Show that the relation

$$f_0 \sim f_1 \iff f_0, f_1 \text{ homotopic as pointed maps}$$

is an equivalence relation.

- $f \sim f$
- $f_0 \sim f_1 \Rightarrow f_1 \sim f_0$
- $f_0 \sim f_1,\ f_1 \sim f_2 \Rightarrow f_0 \sim f_2$.

Def'n Let (X, x_0) be a pointed space. Then as a set,

$$\pi_n(X, x_0)$$

is defined to be the set of homotopy classes of maps

$$(S^n, *) \longrightarrow (X, x_0),$$

where $* \in S^n$ is some choice of basepoint.

Ex If $n=0$,

$$S^0 = \underset{p}{\cdot} \quad \overset{*}{\cdot}\,.$$

Then a map $(S^0, *) \xrightarrow{f} (X, x_0)$ is the choice of some point $x \in X$, $f(p) = x$.

Two maps are homotopic $\Leftrightarrow$ $f(p)$ and $f'(p)$ are in the same connected component.

$\Rightarrow \pi_0(X, x_0)$ is the set of (path-) connected components of X.

Notation For $n \geq 1$, given $f: (S^n, *) \to (X, x_0)$ we write

$$[f] \in \pi_n(X, x_0)$$

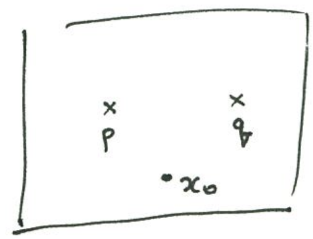
for its homotopy class.

Ex If $n=1$, a map

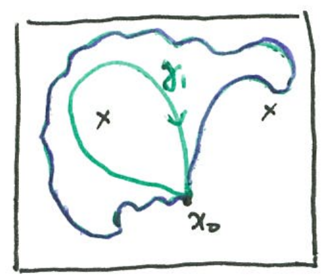
$$f: (S^1, *) \to (X, x_0)$$

is a choice of a loop in X.

Say $X = \mathbb{R}^2 \setminus \{p, q\}$.

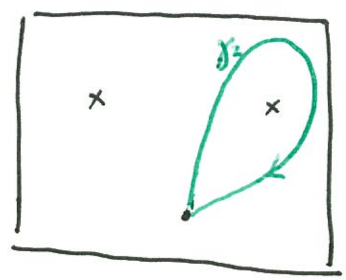


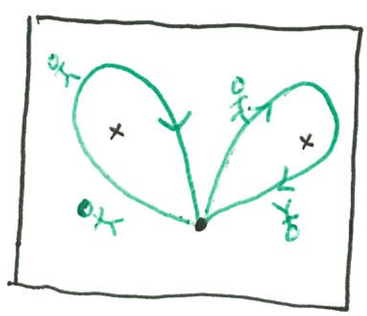
Some examples of loops:



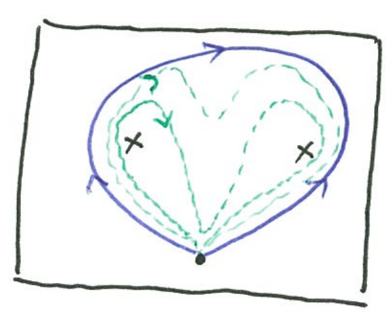
The blue and green loops are homotopic.

γ_1, γ_2 NOT homotopic





"γ_1, then γ_2" is a composition of the two paths.



The blue loop is homotopic to "γ_1, then γ_2".

[The group structure].

Note that a map

$$(S^n, *) \longrightarrow (X, x_0)$$

is the same thing as a map of pairs

$$(D^n, \partial D^n) \to (X, x_0)$$

since $D^n/\partial D^n \cong S^n$.

Further, let $I = [0,1]$ be the unit interval, and

$$I^n \subset \mathbb{R}^n$$

the unit n-cube. This has boundary ∂I^n, and $\exists$ a homeomorphism

$$D^n \xrightarrow{\cong} I^n$$

restricting to

$$\partial D^n \xrightarrow{\cong} \partial I^n.$$

Ex. $I^1 =$ ——, $\partial I^1 =$ $\underset{0}{\cdot}$ $\underset{1}{\cdot}$ $\cong S^0$

$I^2 =$ , $\partial I^2 =$  $\cong S^1$

$I^3 =$  filled in, $\partial I^3 =$  $\cong S^2$

Gives a pair of loops

$$\gamma', \gamma : [0,1] \longrightarrow X$$

s.t. $\gamma(0) = \gamma(1) = x_0$,
$\gamma'(0) = \gamma'(1) = x_0$,

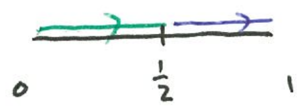
define

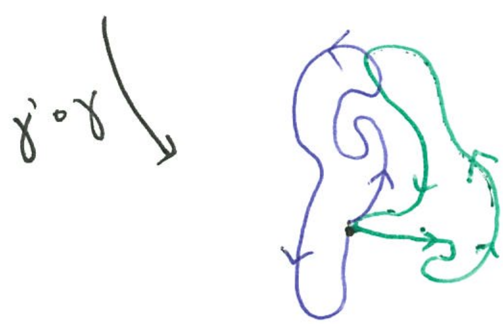
$$\gamma' \circ \gamma : [0,1] \longrightarrow X$$

to be the loop

$$\gamma' \circ \gamma(s) = \begin{cases} \gamma(2s) & s \in [0, \tfrac{1}{2}] \\ \gamma'(2s-1) & s \in [\tfrac{1}{2}, 1] \end{cases}.$$

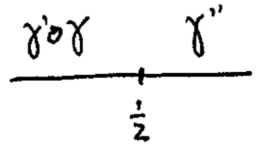
This loop is "do γ, then do γ'".

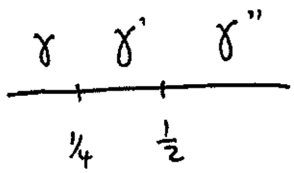


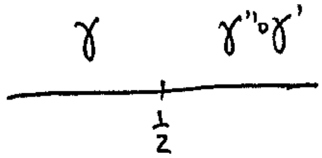


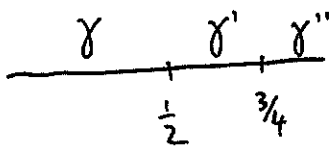
Note the point $s = 1/2$ is sent to x_0.

Is this associative?

$\gamma'' \circ (\gamma' \circ \gamma) =$ 

$=$ 

$(\gamma'' \circ \gamma') \circ \gamma =$ 

$=$ 

These paths are <u>NOT</u> equal!

But they <u>are</u> homotopic.

<u>Prop'n</u> $(\gamma'' \circ \gamma') \circ \gamma \sim \gamma'' \circ (\gamma' \circ \gamma)$.

<u>Pf.</u> $\forall\, t \in [0,1]$, let

$$g_t : I \to I$$

be the map

$$g_t(s) = \begin{cases} (1+t)s & s \in [0, 1/4] \\ s + t/4 & s \in [1/4, 1/2] \\ (1 - \frac{t}{2})s + \frac{t}{2} & s \in [1/2, 1]. \end{cases}$$

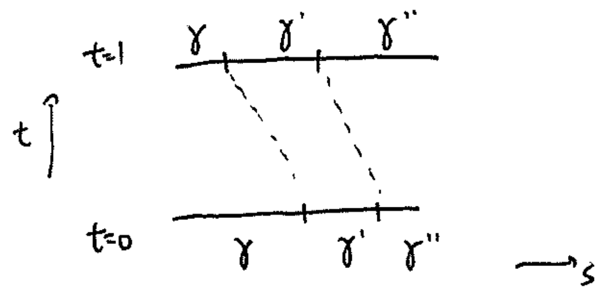
Note $g_0(s) = s$.

And set

~~$F(s,t) := (\gamma'' \circ (\gamma' \circ \gamma))(g_t(s))$~~

$$F(s,t) := ((\gamma'' \circ \gamma') \circ \gamma)(g_t(s)).$$

g_t is a homotopy that re-parametrizes the interval so



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Defn. Let f, f' be maps

$$(I^n, \partial I^n) \longrightarrow (X, x_0).$$

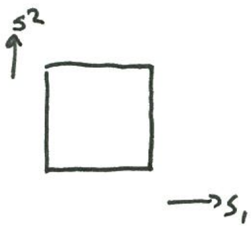
We define a new map

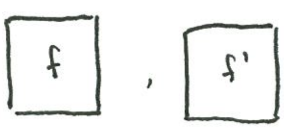
$$f' \circ f : (I^n, \partial I^n) \longrightarrow (X, x_0)$$

by

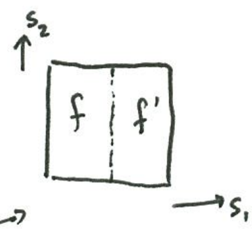
$$f' \circ f(s_1, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & s_1 \in [0, \tfrac{1}{2}] \\ f'(2s_1 - 1, s_2, \dots, s_n) & s_1 \in [\tfrac{1}{2}, 1] \end{cases}.$$

Ex. If $n=2$, draw

$I^2 =$ 

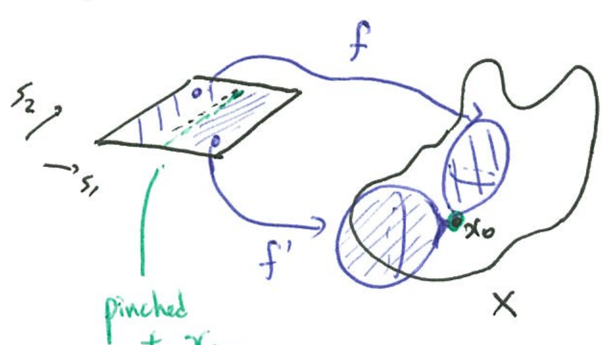
Given 

we have

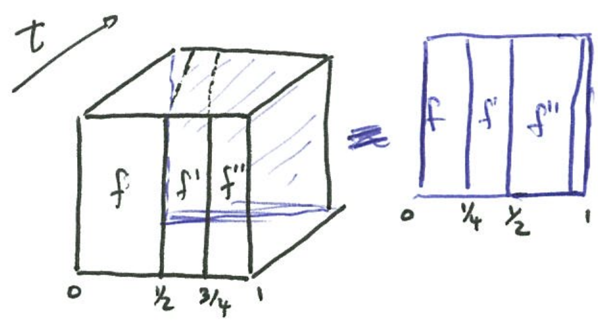
$f' \circ f =$ 

(Note all of the segment $s_1 = \frac{1}{2}$ is sent to x_0.)

This picture means



As before, $f'' \circ (f' \circ f) \neq (f'' \circ f') \circ f$, but these compositions are homotopic!



Defn $\forall\, n \geq 1$, let

$$\pi_n(X, x_0) \times \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0)$$

be the map

$$([f'], [f]) \longmapsto [f' \circ f].$$

← Associative by previous proposition!

Thm $\forall\, n \geq 1$, $\pi_n(X, x_0)$ is a group under the above operation.

If $n \geq 2$, $\pi_n(X, x_0)$ is abelian.

Pf. Unit: The constant map $S^n \xrightarrow{e} x_0$.

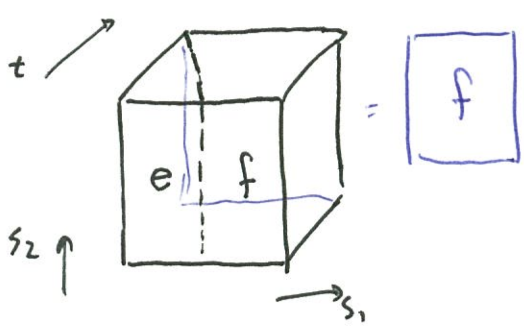
Consider the homotopy

$$h_t(s) = (1 - \tfrac{t}{2})s + \tfrac{t}{2}.$$

The function

$$F((s_1, \dots, s_n), t) = (f \circ e)(h_t s_1, s_2, \dots, s_n)$$

gives a homotopy from f to $f \circ e$:



You can likewise find a homotopy from $(e \circ f)$ to f. Hence

$$[f \circ e] = [e \circ f] = [f].$$

Inverses: Given $f:(I^n, \partial I^n) \to (X, x_0)$,

consider

$$g(s_1, \cdots, s_n) := f(1-s_1, s_2, \cdots, s_n).$$

If $n=1$, this is just "running the path f backwards."

The function

$$H: I^n \times [0,1] \longrightarrow X$$

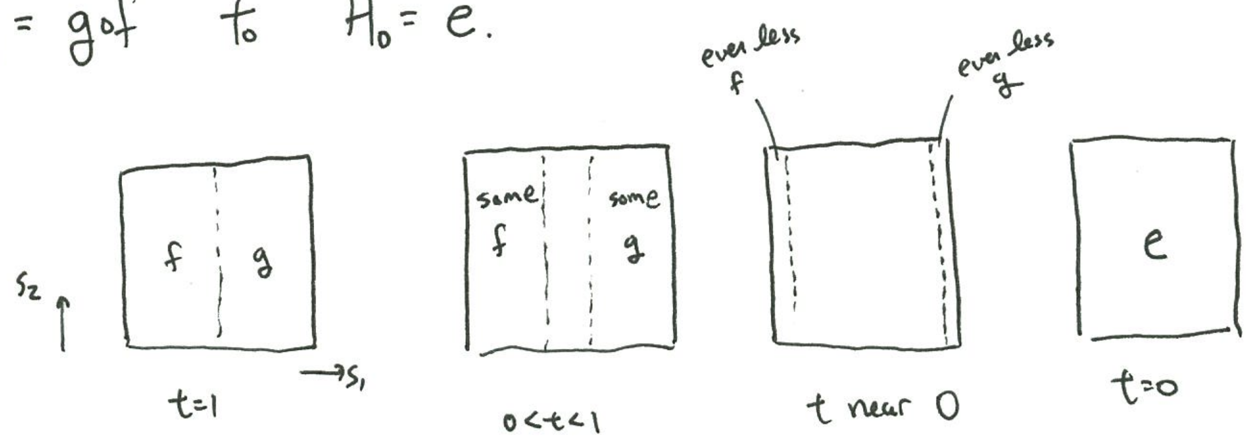
given by

$$H_t(s_1, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots, s_n) & s_1 \in [0, t/2] \\ f(t, s_2, \cdots, s_n) & s_1 \in [t/2, 1-t/2] \\ f(1-2s_1, s_2, \cdots, s_n) & s_1 \in [1-t/2, 1] \end{cases}$$

realizes a homotopy from

$$H_1 = g \circ f \quad \text{to} \quad H_0 = e.$$

since $f(0, s_2, \cdots, s_n) = x_0$.

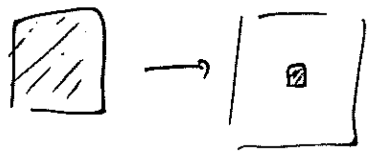


n=1 picture:

Commutativity for $n \geq 2$.

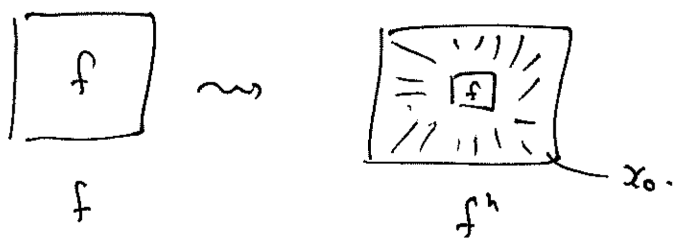
Note that any shrinking ~~map~~ embedding

$$h : I^n \longrightarrow I^n, \quad h(I^n) \subsetneq I^n$$



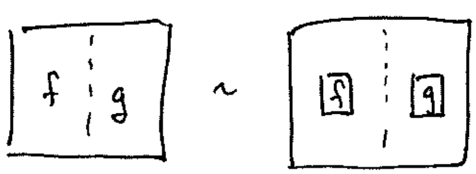
defines a new map $f^h : (I^n, \partial I^n) \longrightarrow (X, x_0)$ by

$$f^h(\vec{s}) = \begin{cases} x_0 & \vec{s} \notin h(I^n) \\ f(h^{-1}(\vec{s})) & \vec{s} \in h(I^n) \end{cases}.$$

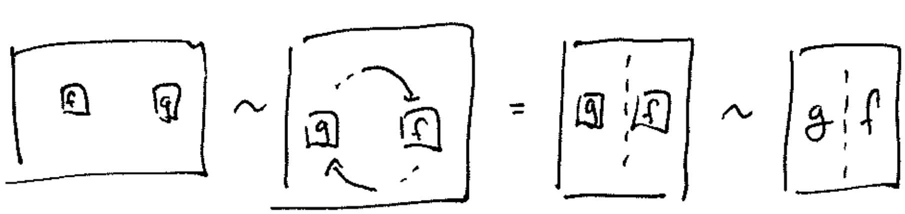


If h is homotopic to $id : I^n \to I^n$, f^h is homotopic to f.

So $[g \circ f] = [g^h \circ f^h]$.



Take a homotopy that slides [f], [g] cubes like hockey pucks



So $g^h \circ f^h \sim f^h \circ g^h$

i.e., $[g \circ f] = [f \circ$