π_n as a functor

objects
- (X, x_0) $\qquad$ $\pi_n(X, x_0)$

morphisms
- $f: (X, x_0) \longrightarrow (Y, y_0)$ $\qquad$ $f_*: \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0)$

$$[g] \longmapsto [f \circ g]$$

- $$\begin{tikzcd} (D^n, \partial D^n) \arrow[r, "g"] \arrow[dr, "f \circ g"'] & (X, x_0) \arrow[d, "f"] \\ & (Y, y_0) \end{tikzcd}$$

On this page, "$\circ$" means actual function composition; NOT the π_n composition.

- $g \sim g' \Rightarrow f \circ g \sim f \circ g'$, so well-defined.

- $\boxed{g \mid g'} \xrightarrow{f \circ} \boxed{f \circ g \mid f \circ g'}$.

~~so~~ So f_* is a group homomorphism.

identity
- Constant map is sent to constant map.

$$(D^n, \partial D^n) \xrightarrow{e} (X, x_0) \xrightarrow{f} (Y, y_0)$$

composition
- $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{f'} (Z, z_0)$ $\qquad$ $\pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, y_0) \xrightarrow{f'_*} \pi_n(Z, z_0)$

$$(f'_* \circ f_*)[g] = [f' \circ (f \circ g)]$$
$$= [(f' \circ f) \circ g]$$
$$= (f' \circ f)_* [g].$$

homotopy invariance
- If f_0 is homotopic to f_1, $(f_0)_* = (f_1)_*$.

(Given $g: (D^n, \partial D^n) \to (X, x_0)$,
then $f_0 \circ g$ is homotopic to $f_1 \circ g$. So $[f_0 \circ g] = [f_1 \circ g]$)

way easier to prove homtpy invariance for π_* than for H_*!

Relative homotopy groups

Last time we defined homotopy groups:

$$\pi_n(X, x_0) := \mathrm{Maps}_*((S^n, *), (X, x_0)) / \text{homotopy}$$

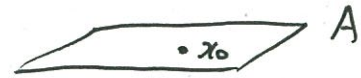
And we saw this was an abelian group.

<u>Rmk</u> How did we get a group out of homology?
We deliberately "linearized" spaces of maps by taking
$\mathbb{Z}\,\mathrm{Maps}(\Delta^n, X)$. Nothing about Δ^n wanted to
intrinsically turn this mapping space into a group,
so we artificially threw in elements like
"minus σ", $\sigma: \Delta^n \to X$.

But for π_n, the group structure simply emerged
from the geometry of based maps. Didn't need
take a free abelian group on anything!

~~Rmk Relative homology was~~

Let $A \subset X$ be a subspace s.t. $x_0 \in A$.



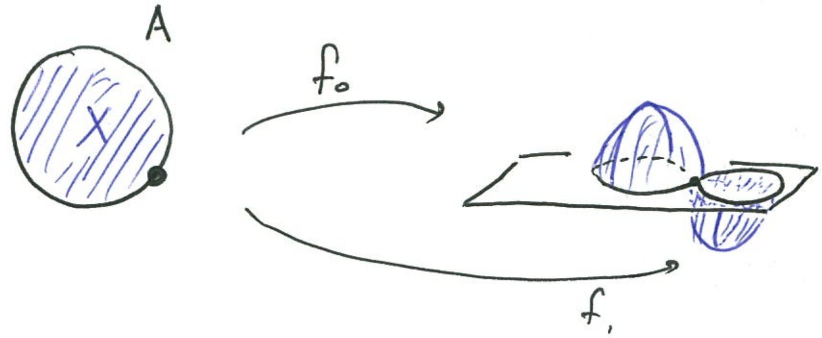
Choose a basepoint $* \in \partial D^n$,

and let $f_0, f_1 : D^n \longrightarrow X$

be maps s.t.

$$f_0(\partial D^n) \subset A \qquad f_1(\partial D^n) \subset A$$

$$f_0(*) = x_0 \qquad f_1(*) = x_0$$



Say $f_0 \sim f_1$ iff $\exists\ F : D^n \times [0,1] \longrightarrow X$

s.t.

- $F(-, 0) = f_0$
- $F(-, 1) = f_1$
- $F(*, t) = x_0 \quad \forall\ t \in [0,1]$
- $F(q, t) \in A \quad \forall\ q \in \partial D^n, t \in [0,1]$.

Defn. The relative homotopy groups of (X, A) are

$$\pi_n(X, A, x_0) := \text{Maps}\big((D^n, \partial D^n, *), (X, A, x_0)\big) \big/ \sim .$$

Here, by

$$\mathrm{Maps}\left((D^n, \partial D^n, *), (X, A, x_0) \right)$$

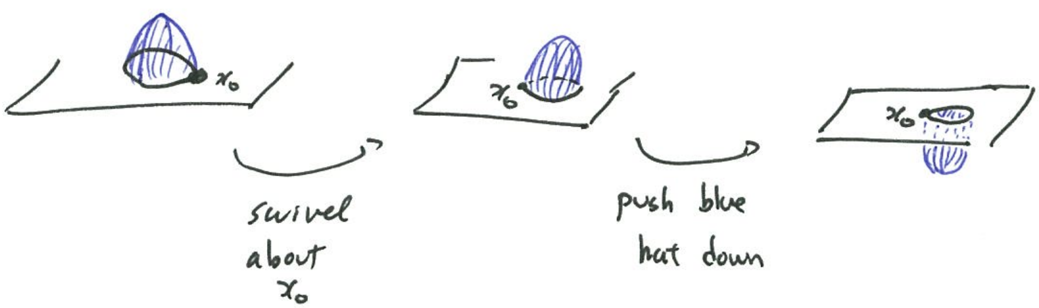
we mean the set of

$$f : D^n \longrightarrow X$$

such that

- $f(\partial D^n) \subset A$
- $f(*) = x_0$.

<u>Ex</u>. The examples of f_0, f_1 I drew are in same equivalence class:



<u>Prop'n</u>. $[f] = [\text{constant map to } x_0]$

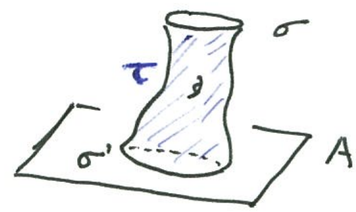
$$\Updownarrow$$

$$\exists\ f'(D) \subset A \text{ s.t. } [f'] = [f].$$

<u>Rmk</u> i.e., If f' sends D^n inside A, the whole map f' can be contracted to $x_0 \in A \subset X$.

Rmk Similar to relative homology.

There, $\sigma \in C_n(X,A)$ has $[\sigma] = [0] \in H_n(X,A)$ iff σ is <u>homologous</u> to $\sigma' \in C_n(A)$.



τ can have all kinds of non-cylindrical topology.

Here, $[f] = [\text{constant map}] \in \pi_n(X, A, x_0)$ iff f is <u>homotopic</u> to $f' \in Maps_*(D^n, A)$.

Pf. Note that D^n deformation retracts to $*$.

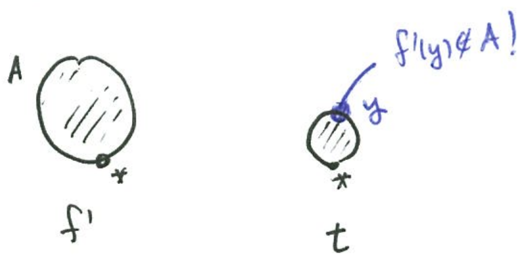
The composite

$$D^n \times [0,1] \xrightarrow{\text{deformation retraction}} D^n \xrightarrow{f'} X$$

gives a homotopy from f' to the constant map. //

Rmk If $f'(D^n)$ is NOT contained in A, this proof fails.

This is because at some time t,



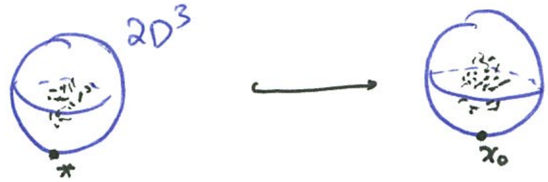
there will exist some $y \in \partial D^n$ s.t. $f_t(y) \notin A$, so $f_t \notin Maps((D^n, \partial D^n, *), (X, A, x_0))$.

__Ex__ Let $X = \mathbb{R}^3$, $A = S^2$.

There's an obvious map

$$f : (D^3, \partial D^3, *) \longrightarrow (X, A, x_0)$$

that just embeds D^3 via standard embedding.



the "⁂" represents the interior of D^3.

$[f] \neq [\text{constant map}]$.

(Why? For instance, f induces a map on relative homology

$$f_* : H_\bullet(D^3, \partial D^3) \longrightarrow H_\bullet(X, A) \cong H_\bullet(S^3 \setminus pt, S^2) \cong H_\bullet(D^3, S^2)$$

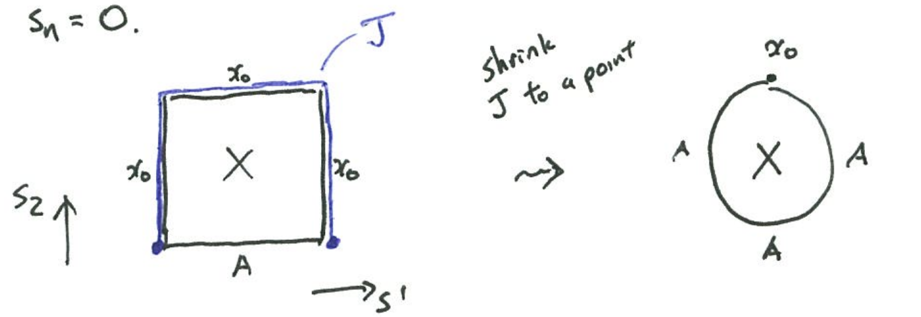
which is an isomorphism on H_3.

But the constant map induces the zero map on H_3.)

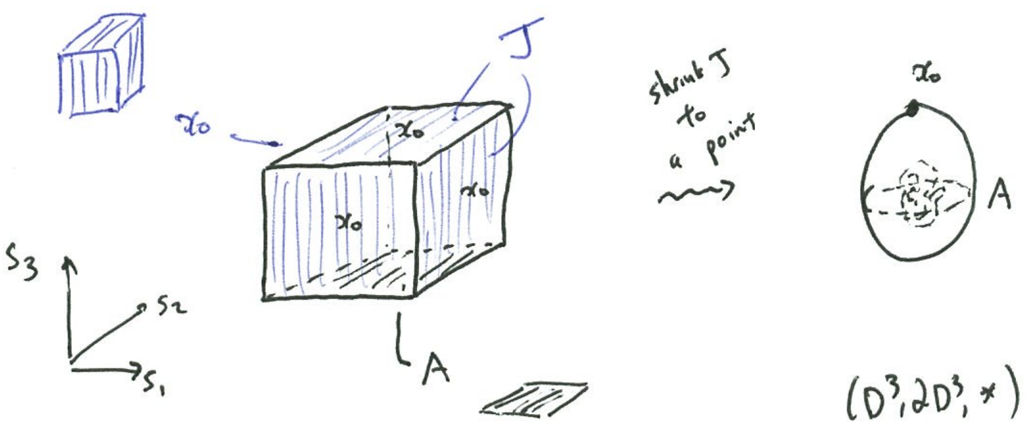
For $n \geq 2$, there's a group structure on $\pi_n(X, A, x_0)$.

Note a map
$$(D^n, \partial D^n, *) \longrightarrow (X, A, x_0)$$
is the same thing as a map
$$(I^n, \partial I^n, J) \longrightarrow (X, A, x_0)$$

where $J \subset \partial I^n$ is what you get when you delete the interior of the face $s_n = 0$.



$(I^2, \partial I^2, J)$ $\qquad$ $(D^2, \partial D^2, *)$.



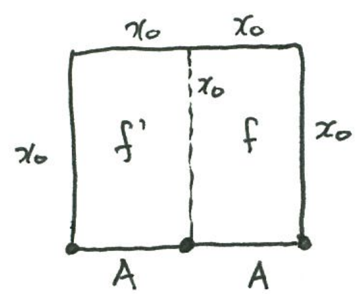
$(I^3, \partial I^3, J)$ $\qquad$ $(D^3, \partial D^3, *)$

So given two maps $f, f' : (I^n, \partial I^n, J) \longrightarrow (X, A, x_0)$

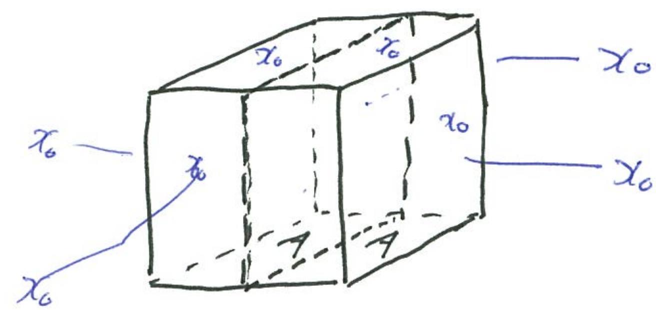
define

$$f' \circ f(s_1, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots, s_n) & s_1 \in [0, \frac{1}{2}] \\ f'(2s_1 - 1, s_2, \cdots, s_n) & s_1 \in [\frac{1}{2}, 1] \end{cases}$$

so we have



$n = 2$



$n = 3$

<u>Rmk</u> Note that given $f : (I^n, \partial I^n, J) \longrightarrow (X, A, x_0)$

the restriction $f\big|_{s_n = 0}$

to the bottom face produces an element of $\pi_{n-1}(A, x_0)$.

Obviously,

$$(f' \circ f)\big|_{s_n = 0} = \left(f'\big|_{s_n = 0}\right) \circ \left(f\big|_{s_n = 0}\right)$$

so we have a homomorphism

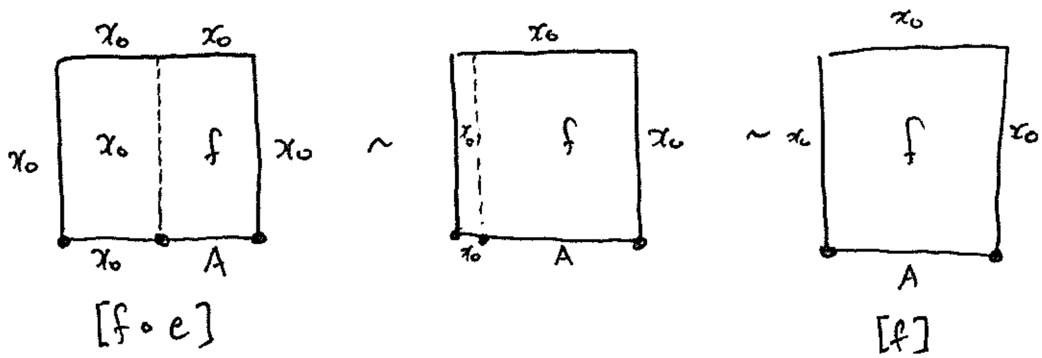
$$\pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0).$$

Isn't this fun? Very natural and clever!

This will be the connecting homomorphisms in a long exact sequence.

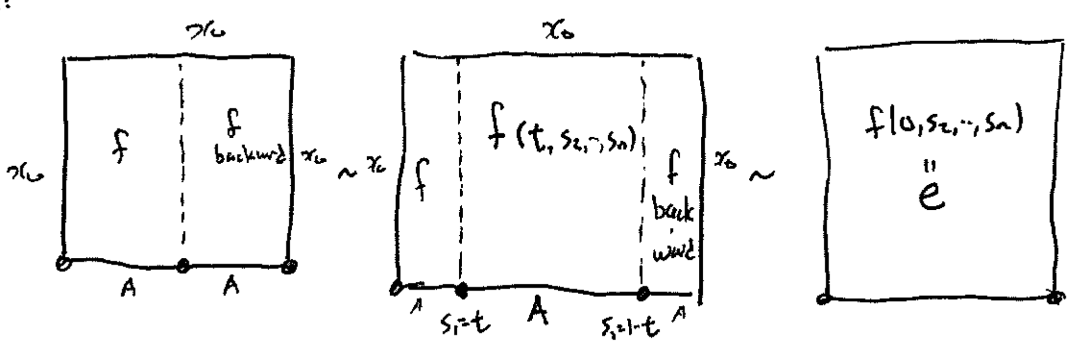
Rmk I got ahead of myself. This operation $f' \circ f$ really defines a group structure on $\pi_n(X, A, x_0)$.

- Identity: e = constant map to x_0.

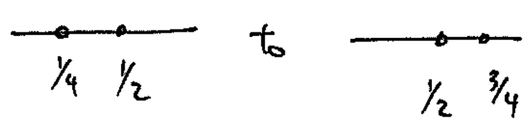


likewise, $[e \circ f] \simeq [f]$.

- Inverses:



- Associativity: Reparametrize



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Rmk A map

$$(X, A, x_0) \longrightarrow (Y, B, y_0)$$

induces a map

$$\pi_n(X, A, x_0) \longrightarrow \pi_n(Y, B, y_0) \quad \forall n.$$

Rmk If $A = \{x_0\}$, then

$$\pi_n(X, A, x_0) \stackrel{?}{=} \pi_n(X, x_0).$$

Consider the composition of continuous maps

$$\underset{\substack{\text{inclusion} \\ \text{of } A}}{(A, x_0, x_0) \xhookrightarrow{i} (X, x_0, x_0)} \underset{\substack{\text{identity} \\ \text{on} \\ X}}{\xhookrightarrow{j}} (X, A, x_0).$$

This induces maps on homotopy groups

$$\begin{array}{ccccc} \pi_n(A, x_0, x_0) & \xrightarrow{i_*} & \pi_n(X, x_0, x_0) & \xrightarrow{j_*} & \pi_n(X, A, x_0). \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \\ \pi_n(A, x_0) & & \pi_n(X, x_0) & & \end{array}$$

Recall the map

$$\pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0)$$

from before.

<u>Thm</u> The sequence

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots$$

is exact.

As we'll see, the proof requires the fact that the pair

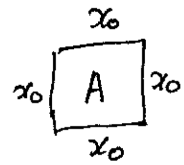
$$(I^n, I^{n-1} = \{\vec{s} \mid s_n = 0\})$$

satisfies the <u>homotopy extension property</u>, or HEP.

Pf. By earlier prop'n, $[f] \in \pi_n(X, A, x_0)$ is equal to $[e]$ iff f is homotopic to a map w/ image completely in A. So

$$\operatorname{Ker} j_* = \text{homotopy classes of maps}$$
$$(D^n, \partial D^n) \to (X, x_0)$$
$$\text{homotopic to maps factoring through } A$$

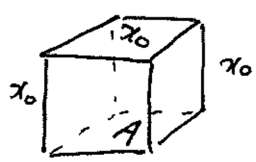
$$= \operatorname{image}(i_*).$$

- Now, $\operatorname{Ker} i_*$ is the set of homotopy classes of maps 

that can be homotoped to the constant map.
This homotopy gives a map

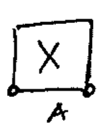
$$F : I^n \times [0,1] \longrightarrow X$$

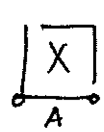
$\operatorname{Ker} i_* = \operatorname{Image} \partial.$

w/ boundary conditions 

which means $[F] \in \pi_{n+1}(X, A, x_0)$. Restricting F to bottom face recovers original map, so $\operatorname{Ker} i_* \subset \operatorname{Image}(\partial)$.

OTOH, a map $F : I^{n+1} \to X$, $[F] \in \pi_{n+1}(X, A, x_0)$ gives a homotopy from the bottom face to the constant map, so $\operatorname{Ker} i_* \supset \operatorname{Image}(\partial)$.

- $\operatorname{Ker} \partial =$ maps  s.t.  can be homotoped to constant map.

By HEP, we get a homotopy from  to , so $\operatorname{Ker} \partial \subset \operatorname{Image}(j)$.

On the other hand, any $[f] \in \operatorname{image}(j)$ looks like  so its restriction to the bottom face is constant. $\operatorname{Ker} \partial \supset \operatorname{Image}(j)$. //

Remark. The LES ends w/

$$\pi_1(X, x_0) \longrightarrow \pi_1(X, A, x_0) \xrightarrow{\partial_1} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0) \xrightarrow{j_*} \pi_0(X, A, x_0) \longrightarrow *.$$

Only $\pi_1(X, x_0)$ is a group, the rest of these are just sets!

Really, pointed sets: [constant map] is a distinguished element in each.

So define ~~$\ker(f) = \{$~~

$$\ker(\partial_1) = \{ [f] \mid \partial_1 [f] = [\text{constant map}] \}$$
$$\ker(i_*) = \{ [f] \mid i_* [f] = [\text{constant map}] \}$$

and define $\pi_0(X, A, x_0)$ to be the set of path-connected components of X that never intersect A.