Recall the map

$$\pi_n(X,A,x_0) \xrightarrow{\partial_n} \pi_{n-1}(A,x_0)$$

from before.

__Thm__ The sequence

$$\cdots \to \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial_n} \pi_{n-1}(A,x_0) \to \cdots$$

is exact.

As we'll see, the proof requires the fact that the pair

$$\left(I^n,\ I^{n-1} = \{\vec{s} \mid s_n = 0\}\right)$$

satisfies the homotopy extension property, or HEP.

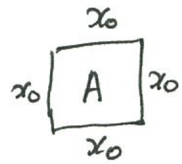
<u>Pf.</u> By earlier prop'n, $[f] \in \pi_n(X, A, x_0)$ is equal to $[e]$ iff f is homotopic to a map w/ image completely in A. So

$\operatorname{Ker} j_* =$ homotopy classes of maps

$$(D^n, \partial D^n) \longrightarrow (X, x_0)$$

homotopic to maps factoring through A

$= \text{image}(i_*)$.



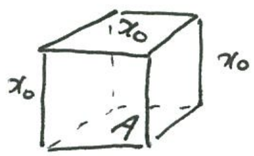
- Now, $\operatorname{Ker} i_*$ is the set of homotopy classes of maps that can be homotoped to the constant map.

This homotopy gives a map

$$F : I^n \times [0,1] \longrightarrow X$$

$\operatorname{Ker} i_* = \operatorname{Image} \partial$.

w/ boundary conditions



which means $[F] \in \pi_{n+1}(X, A, x_0)$. Restricting F to bottom face recovers original map, so $\operatorname{Ker} i_* \subset \operatorname{Image}(\partial)$.

OTOH, a map $F : I^{n+1} \to X$, $[F] \in \pi_{n+1}(X, A, x_0)$ gives a homotopy from the bottom face to the constant map, so $\operatorname{Ker} i_* \supset \operatorname{Image}(\partial)$.

- $\operatorname{Ker} \partial =$ maps $\underset{A}{\boxed{X}}$ s.t. $\underset{A}{\text{—}}$ can be homotoped to constant map.

By HEP, we get a homotopy from $\underset{A}{\boxed{X}}$ to $\underset{x_0}{\boxed{X}}$, so $\operatorname{Ker} \partial \subset \operatorname{Image}(j)$.

On the other hand, any $[f] \in \operatorname{image}(j)$ looks like $\underset{x_0}{\boxed{X}}$ so its restriction to the bottom face is constant. $\operatorname{Ker} \partial \supset \operatorname{Image}(j)$. //

Remark. The LES ends w/

$$\pi_1(X,x_0) \longrightarrow \pi_1(X,A,x_0) \xrightarrow{\partial_1} \pi_0(A,x_0) \xrightarrow{i_*} \pi_0(X,x_0) \xrightarrow{j_*} \pi_0(X,A,x_0) \longrightarrow *.$$

Only $\pi_1(X,x_0)$ is a group, the rest of these are just sets!

Really, pointed sets: [constant map] is a distinguished element in each.

So define ~~Ker(f) = {~~

$$\ker(\partial_1) = \{ [f] \mid \partial_1[f] = [\text{constant map}] \}$$

$$\ker(i_*) = \{ [f] \mid i_*[f] = [\text{constant map}] \}$$

and define $\pi_0(X,A,x_0)$ to be the set of path-connected components of X that never intersect A.

Basepoint change

Propn. Let $\gamma : [0,1] \to X$ be a path from x_0 to x_1.

Then "radial conjugation" by γ induces a group isomorphism

$$\pi_n(X, x_0) \cong \pi_n(X, x_1) \quad \forall\ n \geq 1.$$

Cor. If X is path-connected,

$$\pi_n(X, x_0) \cong \pi_n(X, x_1) \quad \forall\ n$$

$\forall\ x_0, x_1 \in X$.

Rmk. We'll also see that this isomorphism depends only on the homotopy class of γ. So by taking $x_0 = x_1$, we have

Cor. There is a group homomorphism

$$\pi_1(X, x_0) \longrightarrow \mathrm{Aut}(\pi_n(X, x_0)) \quad \forall\ n \geq 1.$$

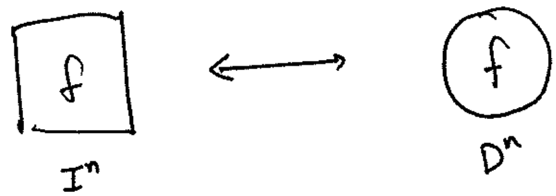
Because of the independence of base point, we may write

$$\pi_n(X) := \pi_n(X, x_0)$$

when X is path-connected.

Remark: The image of this homomorphism is a good indication of how complicated a space is; for instance, if it's trivial, H_n is fin. generated iff π_n is. And a map inducing $\cong$ on H_* is a homtpy equivalence for such spaces.

Choose some homeomorphism $D^n \cong I^n$.

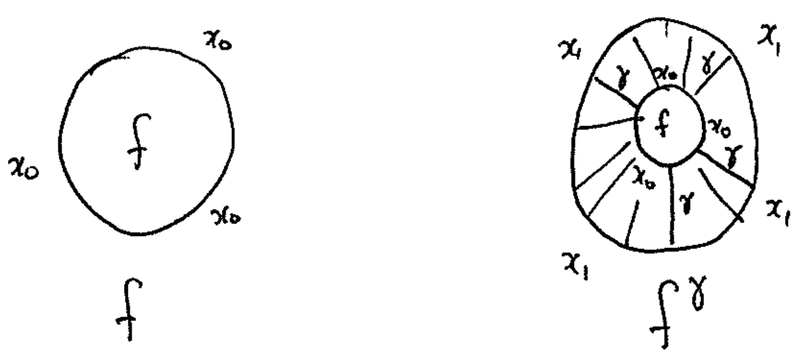


Let $\gamma : [0,1] \longrightarrow X$ be the path from x_0 to x_1.

Given $f : (D^n, \partial D^n) \longrightarrow (X, x_0)$
define $f^\gamma : (D^n, \partial D^n) \longrightarrow (X, x_1)$ by

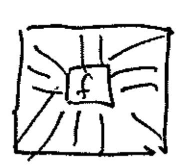
$$f^\gamma(\vec{v}, r) = \begin{cases} f(\vec{v}, 2r) & r \in [0, \tfrac{1}{2}] \\ \gamma(2r-1) & r \in [\tfrac{1}{2}, 1] \end{cases}$$

where $\vec{v} \in \partial D^n = S^{n-1}$, and $(\vec{v}, r) \in D^n$ is the point r units from the origin in the $\vec{v}$ direction.

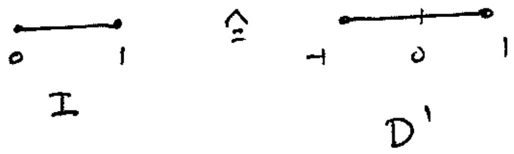


In terms of squares,

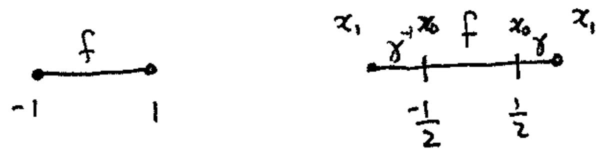
$f =$ 

$f^\gamma =$ 

Ex $n=1$.



Then



as a picture in X,

f $\qquad\qquad$ f^{γ}

Pf of Prop'n We'll show

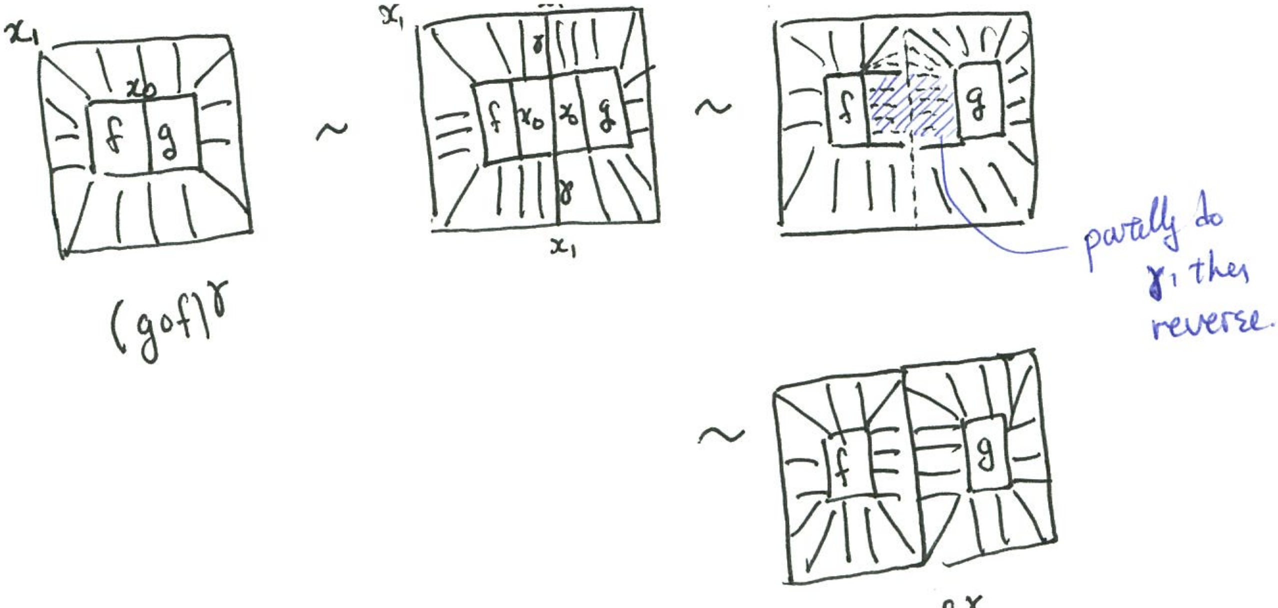
(1) $[(g\circ f)^{\gamma}] = [g^{\gamma}\circ f^{\gamma}]$ $\qquad$ Each γ defines a homomorphism $\pi_n(X,x_0)\to\pi_n(X,x_1)$.

(2) $[f^{\gamma_2\circ\gamma_1}] = [(f^{\gamma_1})^{\gamma_2}]$ $\qquad$ Here, $\gamma_1(0)=x_0$, $\gamma_1(1)=x_1$, $\gamma_2(0)=x_1$, $\gamma_2(1)=x_2$.

(3) $[f^{e}] = [f]$, e constant path.

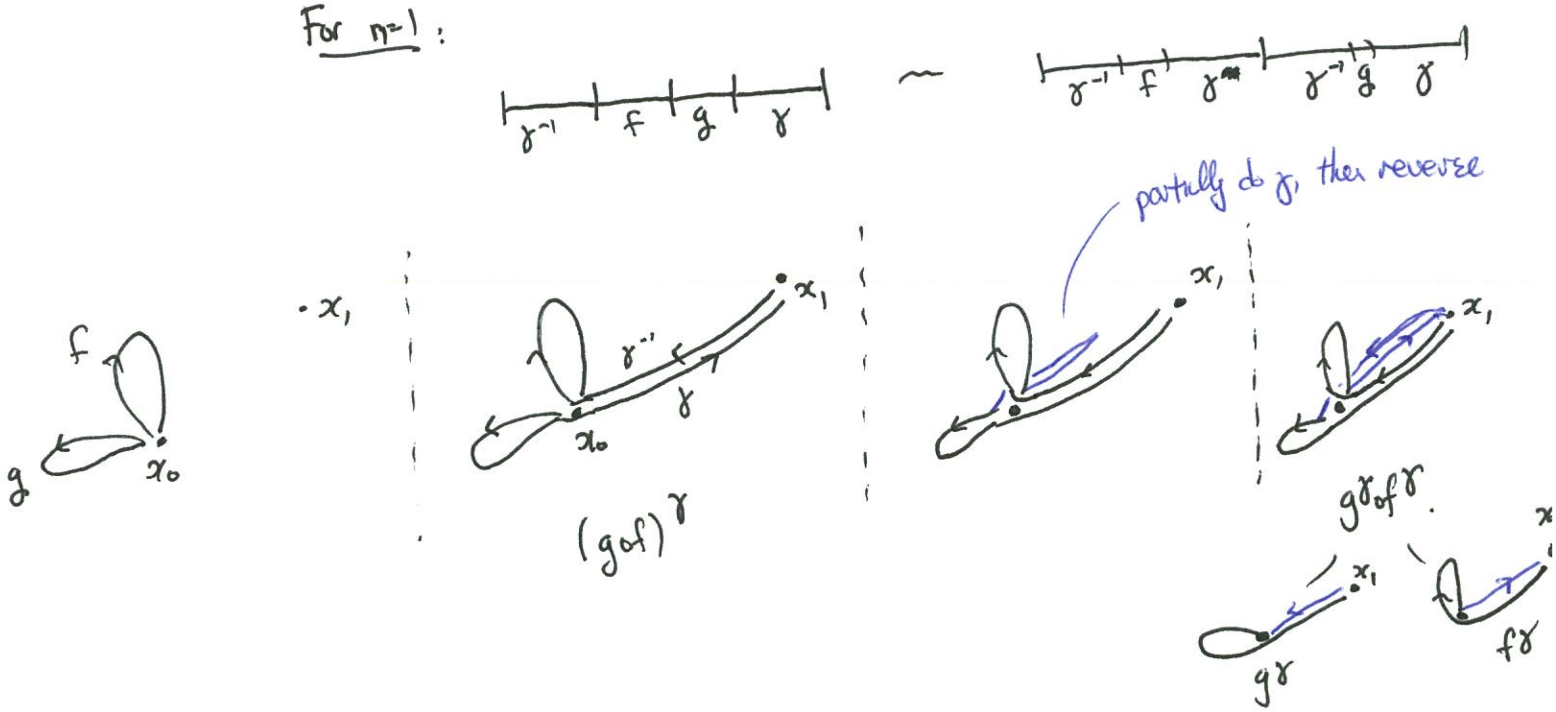
Note (2), (3) show that the map "conjugate by γ^{-1}" is inverse to "conjugate by γ".

(1)

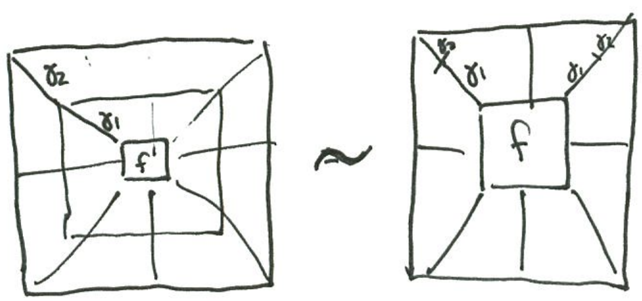


$(g\circ f)^\gamma$ $\qquad$ $g^\gamma \circ f^\gamma$

For $n=1$:



(2)



$(f^{\gamma_1})^{\gamma_2} \sim f^{\gamma_2\circ\gamma_1}$

(3) exercise.

Some remarks:

What is the relation(ship) between $\pi_n(X/A)$ and $\pi_n(X, A, x_0)$?

Opens the door to some philosophy:

	homology	homotopy groups
Unions of spaces	easy (Mayer-Vietoris)	harder ($\cong$ on low dimensions, $\pi_\cdot(A, B\cap A) \to \pi_\cdot(X, B)$.)
quotients of spaces	easy ($\tilde{H}_n(X,A) \cong \tilde{H}_n(X/A)$)	harder ($\cong$ on low dimensions $\pi_\cdot(X,A) \to \pi_\cdot(X/A)$)
products of spaces	harder (Künneth theorem) (later in the course)	easy ($\pi_n(X\times Y) \cong \pi_n(X) \times \pi_n(Y)$)
Fibrations (later in the course)	harder (Serre spectral sequence)	easy (Long exact sequences)

As it turns out,

unions and quotients

are examples of "colimits."

products and fibrations

are examples of "limits."

The philosophy is:

homology behaves well with respect to colimits.

π_n " limits.

Compression Lemma

Now that we've got some examples of invariants, and we know they're all invariant under homotopies, what techniques can we develop to homotope maps $f: X \to Y$ to something more convenient?

$f_0: X \to Y$ $\xrightarrow{\text{homotope}}$ $f_1: X \to Y$

crazy-looking

Ah, much nicer!

It's hard to describe what makes a map "nice" w/out some structures on X and Y.

The most convenient structure (as we'll see) is that of a CW complex.

Lemma Let (X,A) be a CW pair.

Let (Y,B), $B \neq \emptyset$, be a pair of spaces.

(Not nec. CW.)

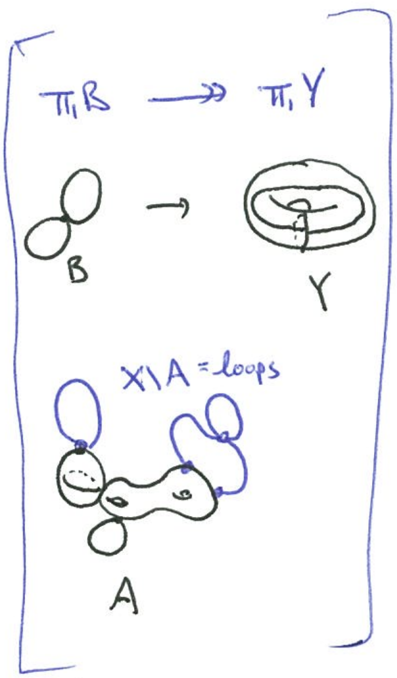
Assume: $\forall\, n$ s.t. $X - A$ has an n-cell,

$$\pi_n(Y, B, y_0) = 0.$$

Then: Every map of pairs

$$f : (X,A) \to (Y,B)$$

is homotopic rel A to a map $X \to B$.



example.

Ex If Y deformation retracts onto B, then

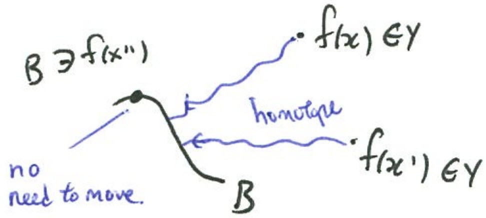
$$B \hookrightarrow Y$$

is a homotopy equivalence, so $\pi_n(Y, B, y_0) = 0$ $\forall n$ by the LES. So regardless of the cells in $X \setminus A$, any map $(X,A) \to (Y,B)$ can be homotoped to lie inside B. (As we know.)

Ex If $X \setminus A$ only has 1-cells, and $\pi_1 B \to \pi_1 Y$ is a surjection, then any $(X,A) \to (Y,B)$ can be squeezed to lay inside B.

<u>Pf</u> For $n=0$, if $\pi_n(Y, B, y_0) = 0$, we mean every point in Y is connected by some path to a point in B.

So we can certainly homotope $f|_{X^0}$ to lie inside B.



This homotopy extends to a homotopy on all of X since (X, X^0) has HEP by homework.

By induction, assume f is homotopic to a map taking X^{k-1} to B, rel A.

If X^k has no k-cell outside of A, we can start trying to homotope X^{k+1}.

~~Since the~~

Otherwise, we know $\pi_k(Y, B, y_0) = 0$. By LES of relative homotopy groups,

$$\pi_k(B, y_0) \longrightarrow \pi_k(Y, y_0)$$

must be a surjection.

So any $(D^k, \partial D^k) \to (Y, y_0)$ is homotopic rel ∂D^k to a map w/ image in B.

So for any k-cell in X,

$$\cancel{(D^k, \partial D^k) \longrightarrow (X^k, X^{k-1} \cup A)}$$

$$(D^k, \partial D^k) \longrightarrow (X^k, X^{k-1}) \longrightarrow (Y, B)$$

is homotopic to a map w/ image in B.

Doing this for all k-cells, we've homotoped the map on X^k to one contained in B.

By HEP for CW pairs (like (X, X^k)) this homotopy defines a homotopy on all of X.

So

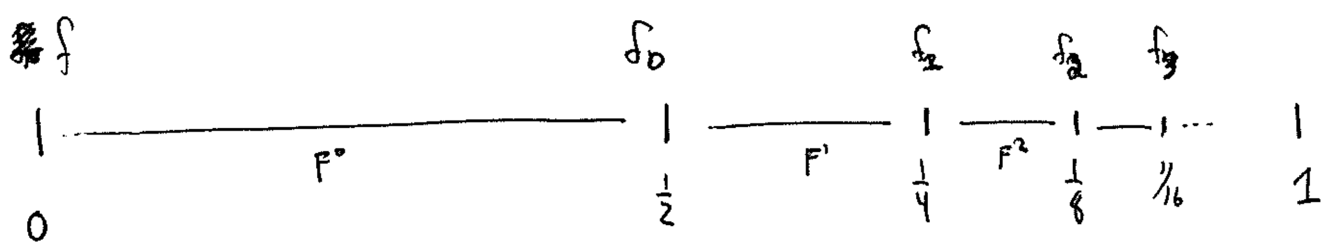
$$f \overset{F^0}{\sim} f^0 \overset{F^1}{\sim} f^1 \overset{F^2}{\sim} f^2 \sim \text{-----}$$

taking

$f^0(X^0) \subset B \qquad f^1(X^1) \subset B \qquad f^2(X^2) \subset B$

is an infinite sequence of homotopies.

Doing the k^{th} homotopy on fast-forward on the interval $\left[1-\frac{1}{2^k}, 1-\frac{1}{2^{k+1}}\right]$



we get a homotopy from f to the desired map.

(See homework.)

//

Corollary If $\pi_n(X, A, x_0) = 0 \ \forall \ n \le N$,

then $H_n(X, A) = 0 \ \forall \ n \le N$.

Pf. Choose $\alpha \in C_n(X, A)$ such that $\partial_n \alpha = 0$.

(This means $\alpha = \sum n_i \sigma_i$ for some maps $\sigma_i : \Delta^n \to X$,

and $\partial_n^X \alpha = \sum n_i \partial \sigma_i \in C_n(A)$.)

Well, the set $\{\sigma_i\}$ is a subset of $\mathrm{Maps}(\Delta^n, X)$.

So they generate a sub-semisimplicial set:

$$
\begin{tikzcd}
\emptyset \arrow[r, hook] \arrow[d, shift left, shift right] & \mathrm{Maps}(\Delta^{n+1}, X) \arrow[d, shift left, shift right] \\
\{\sigma_i\} \arrow[r, hook] \arrow[d, shift left, shift right] & \mathrm{Maps}(\Delta^n, X) \arrow[d, shift left, shift right] \\
\{d_j \sigma_i\} \arrow[r, hook] \arrow[d, shift left, shift right] & \mathrm{Maps}(\Delta^{n-1}, X) \arrow[d, dotted] \\
\{d_j d_{j'} \sigma_i\} \arrow[d, dotted] & \vdots \arrow[d, Rightarrow] \\
\{\text{vertices of } \sigma_i\} \arrow[r, hook] & \mathrm{Maps}(\Delta^0, X)
\end{tikzcd}
$$

Let K be the geometric realization. It's a CW complex.

Now, $\partial\alpha \in C_n(A)$,

$$\partial\alpha = \sum m_i \tau_i, \qquad \tau_i : \Delta^{n-1} \to A.$$

Let L be the geometric realization of the semisimplicial set determined by Since $\tau_i \subset \{d_j \sigma_i\}$, we have a CW pair

$$(K, L).$$

Since $K \setminus L$ has only n-cells, the map

$$(K, L) \xrightarrow{j} (X, A)$$

Note $\sum n_i \sigma_i$ define a homology class in $H_n(K,L)$ and hence by definition, $[\alpha] \in \text{image}(j)$.

can be homotoped to a map w/ image in A.

So we have a homotopy-commutative diagram

$$\begin{tikzcd}
(K, L) \arrow[r] \arrow[dr, "j"'] & (A, A) \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\
 & (X, A)
\end{tikzcd}$$

so applying H_*, j_* must be the zero map.

This shows any $[\alpha] \in H_n(X,A)$ must be in the image of a zero map; hence $[\alpha] = 0$.

//

Before proving ~~the~~ more theorems and the lemmas, some exercises:

<u>Exer</u> (a) Suppose a space M deformation retracts onto a subspace X. This means $\exists$

$$F: M \times [0,1] \longrightarrow M$$

$$(m,0) \longmapsto m$$

$$(m,1) \in X$$

$$(x,t) \longmapsto x \quad \forall t, \ \forall x \in X.$$

Show ~~[illegible]~~ X is homotopy equiv to M.

(b) If $X \longrightarrow M$ and $M \longrightarrow Y$ are homotopy equivalences, show the composition

$$X \longrightarrow Y$$

is a homotopy equivalence.

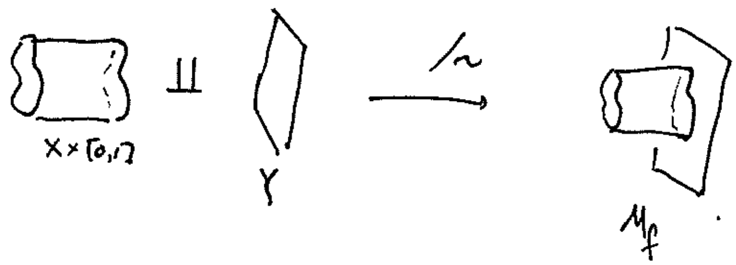
(c) If $f': X \longrightarrow Y$ is a hmtpy equivalence and f' is homotopic to f, show f is a homotopy equivalence.

Mapping Cylinders

Defn Given a continuous map $f: X \to Y$, let

$$M_f := Y \amalg X \times [0,1] \Big/ (x,1) \sim f(x)$$

be the <u>mapping</u> cylinder of f.

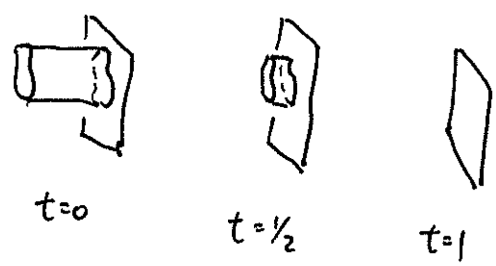


Rmk • M_f deformation retracts onto Y.

$$M_f \times [0,1]_t \longrightarrow M_f$$

$$((x,s),t) \longmapsto (x, s+t-ts)$$

$$y \longmapsto y$$



• If $X \hookrightarrow M_f$, $x \mapsto (x,0)$ is a homotopy equivalence, so is f.

$$\begin{tikzcd} X \arrow[r, hook, "\text{inclusion}"] \arrow[dr, "f"'] & M_f \arrow[d, "\simeq"', "\text{retraction}"] \\ & Y \end{tikzcd}$$

Corollary If $f: X \to Y$ induces an isomorphism

$$\pi_n(X, x_0) \xrightarrow{\sim} \pi_n(Y, y_0) \qquad n \geq 0$$

$\forall\, x_0 \in X$, then

$$f_*: H_n(X) \to H_n(Y)$$

is an isomorphism $\forall\, n \geq 0$.

Pf. Assume X, Y connected. Since the diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"', "\sim"] & M_f \arrow[d, "\wr", "\text{retract}"'] \\ & Y \end{tikzcd}$$

commutes,

$$\begin{tikzcd} \pi_n(X) \arrow[r, "i_*"] \arrow[dr, "\wr"'] & \pi_n(M_f) \arrow[d, "\wr"] \\ & \pi_n(Y) \end{tikzcd}$$

commutes, so i_* is an isomorphism.

$\Rightarrow$ $\pi_n(M_f, X, x_0) \cong 0$ by LES of relative homotopy groups.

$\Rightarrow$ $H_n(M_f, X) \cong 0$ $\forall n$ by previous corollary

$\Rightarrow$ The maps $H_n(X) \xrightarrow{i_*} H_n(M_f)$ are all $\cong$ by LES on relative homology

$\Rightarrow$ $\begin{tikzcd} H_n(X) \arrow[r, "i_*"] \arrow[dr, "f_*"'] & H_n(M_f) \arrow[d, "\wr"] \\ & H_n(Y) \end{tikzcd}$, shows f_* is $\cong$ $\forall\, n$. //