Compression Lemma

Now that we've got some examples of invariants, and we know they're all invariant under homotopies, what techniques can we develop to homotope maps $f: X \to Y$ to something more convenient?

$$f_0 : X \to Y \quad \xrightarrow{\text{homotope}} \quad f_1 : X \to Y$$

crazy-looking

Ah, much nicer!

It's hard to describe what makes a map "nice" w/out some structures on X and Y.

The most convenient structure (as we'll see) is that of a CW complex.

Lemma Let (X,A) be a CW pair.

Let (Y,B), $B \neq \emptyset$, be a pair of spaces.

(Not nec. CW.)

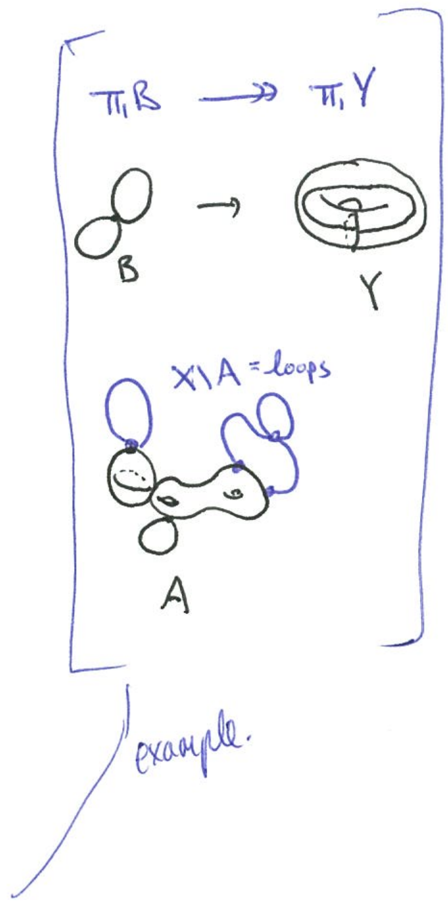
Assume: $\forall\, n$ s.t. $X - A$ has an n-cell,

$\pi_n(Y, B, y_0) = 0.$

Then: Every map of pairs

$f: (X,A) \to (Y,B)$

is homotopic rel A to a map $X \to B$.



Ex If Y deformation retracts onto B, then

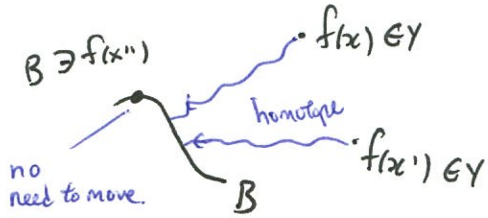
$B \hookrightarrow Y$

is a homotopy equivalence, so $\pi_n(Y, B, y_0) = 0$ $\forall n$ by the LES. So regardless of the cells in $X \setminus A$, any map $(X,A) \to (Y,B)$ can be homotoped to lie inside B. (As we know.)

Ex If $X \setminus A$ only has 1-cells, and $\pi_1 B \to \pi_1 Y$ is a surjection, then any $(X,A) \to (Y,B)$ can be squeezed to lay inside B.

Pf. For $n=0$, if $\pi_n(Y,B,y_0)=0$, we mean every point in Y is connected by some path to a point in B.

So we can certainly homotope $f|_{X^0}$ to lie inside B.



This homotopy extends to a homotopy on all of X since (X, X^0) has HEP by homework.

By induction, assume f is homotopic to a map taking X^{k-1} to B, rel A.

If X^k has no k-cell outside of A, we can start trying to homotope X^{k+1}.

~~Since~~

Otherwise, we know $\pi_k(Y,B,y_0)=0$. ~~By LES of relative homotopy groups,~~

$$\cancel{\pi_k(B,y_0) \longrightarrow \pi_k(Y,y_0)}$$

~~must be a surjection.~~

~~So any $(D^k,\partial D^k) \to (Y,y_0)$ is homotopic rel ∂D^k to a map w/ image in B.~~

So for any k-cell in X,

$$\cancel{(D^k,\partial D^k) \longrightarrow (X^k, X^{k-1}\cup A)}$$

$$(D^k,\partial D^k) \longrightarrow (X^k, X^{k-1}) \longrightarrow (Y,B)$$

is homotopic to a map w/ image in B. (By definition of relative homotopy groups)

Doing this for all k-cells, we've homotoped the map on X^k to one contained in B.

By HEP for CW pairs (like (X, X^k)) this homotopy defines a homotopy on all of X.

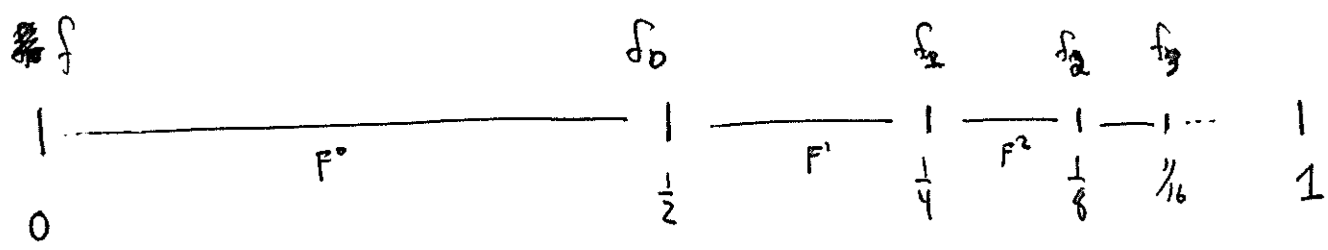
So

$$f \overset{F^0}{\sim} f^0 \overset{F^1}{\sim} f^1 \overset{F^2}{\sim} f^2 \sim \cdots$$

taking $f^0(X^0) \subset B$ $\quad f^1(X^1) \subset B$ $\quad f^2(X^2) \subset B$

is an infinite sequence of homotopies.

Doing the k^{th} homotopy on fast-forward on the interval $[1-\frac{1}{2^k}, 1-\frac{1}{2^{k+1}}]$



we get a homotopy from f to the desired map.

(See homework.) //

Corollary If $\pi_n(X, A, x_0) = 0 \ \forall \ n \leq N$,

then $H_n(X, A) = 0 \ \forall \ n \leq N$.

Pf. Choose $\alpha \in C_n(X, A)$ such that $\partial_n \alpha = 0$.

(This means $\alpha = \sum n_i \sigma_i$ for some maps $\sigma_i : \Delta^n \to X$,

and $\partial_n^X \alpha = \sum n_i \partial \sigma_i \in C_n(A)$.)

Well, the set $\{\sigma_i\}$ is a subset of $Maps(\Delta^n, X)$.

So they generate a sub-semisimplicial set :

$$\begin{tikzcd}
\emptyset \arrow[r, hook] \arrow[d, Rightarrow] & Maps(\Delta^{n+1}, X) \arrow[d, Rightarrow] \\
\{\sigma_i\} \arrow[r, hook] \arrow[d, Rightarrow] & Maps(\Delta^n, X) \arrow[d, Rightarrow] \\
\{d_j \sigma_i\} \arrow[r, hook] \arrow[d, Rightarrow] & Maps(\Delta^{n-1}, X) \arrow[ddd, dotted] \\
\{d_j d_{j'} \sigma_i\} \arrow[d, Rightarrow] & \\
\vdots \arrow[d, Rightarrow] & \\
\{\text{vertices of } \sigma_i\} \arrow[r, hook] & Maps(\Delta^0, X)
\end{tikzcd}$$

Let K be the geometric realization. It's a CW complex.

Now, $\partial\alpha \in C_n(A)$,

$$\partial\alpha = \sum m_i \tau_i, \qquad \tau_i : \Delta^{n-1} \to A.$$

Let L be the geometric realization of the semisimplicial set determined by τ_i. Since $\tau_i \subset \{d_j\sigma_i\}$, we have a CW pair

$$(K, L).$$

Since $K \setminus L$ has only n-cells, the map

$$(K, L) \xrightarrow{j} (X, A)$$

Note $\sum n_i\sigma_i$ define a homology class in $H_n(K,L)$ and hence by definition, $[\alpha] \in \text{image}(j)$.

can be homotoped to a map w/ image in A.

So we have a homotopy-commutative diagram

$$\begin{tikzcd}
(K, L) \arrow[r] \arrow[dr, "j"'] & (A, A) \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\
& (X, A)
\end{tikzcd}$$

so applying H_*, j_* must be the zero map.

This shows any $[\alpha] \in H_n(X,A)$ must be in the image of a zero map; hence $[\alpha] = 0$.

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