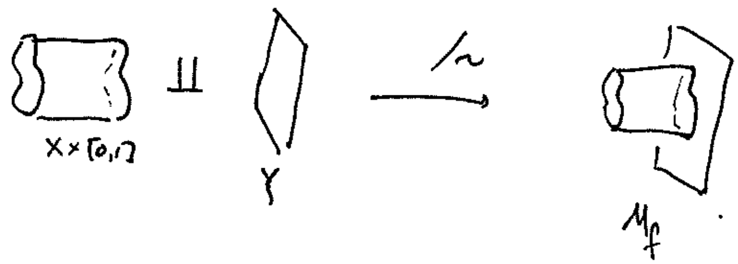
Mapping Cylinders

Defn Given a continuous map $f: X \to Y$, let

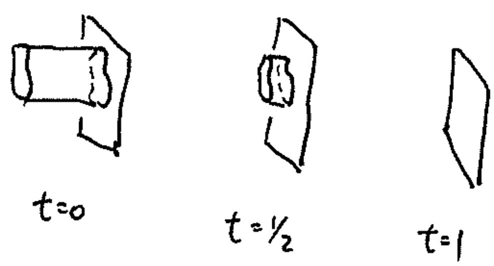
$$M_f := Y \amalg X\times[0,1] \Big/ (x,1)\sim f(x)$$

be the <u>mapping</u> cylinder of f.



Rmk • M_f deformation retracts onto Y.

$$M_f \times [0,1]_t \longrightarrow M_f$$
$$((x,s),t) \longmapsto (x, s+t-ts)$$
$$y \longmapsto y$$



• If $X \hookrightarrow M_f$, $x \mapsto (x,0)$ is a homotopy equivalence, so is f.

$$\begin{tikzcd} X \arrow[r, hook, "\text{inclusion}"] \arrow[dr, "f"'] & M_f \arrow[d, "\text{retraction}", "\wr"'] \\ & Y \end{tikzcd}$$

Corollary If $f: X \to Y$ induces an isomorphism

$$\pi_n(X, x_0) \xrightarrow{\sim} \pi_n(Y, y_0) \qquad n \geq 0$$

$\forall\, x_0 \in X$, then

$$f_*: H_n(X) \to H_n(Y)$$

is an isomorphism $\forall\, n \geq 0$.

Pf. Assume X, Y connected. Since the diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "\sim", "f"'] & M_f \arrow[d, "\wr", "\text{retract}"'] \\ & Y \end{tikzcd}$$

Commutes,

$$\begin{tikzcd} \pi_n(X) \arrow[r, "i_*"] \arrow[dr, "\wr"'] & \pi_n(M_f) \arrow[d, "\wr"] \\ & \pi_n(Y) \end{tikzcd}$$

Commutes, so i_* is an isomorphism.

$\Rightarrow$ $\pi_n(M_f, X, x_0) \cong 0$ by LES of relative homotopy groups.

$\Rightarrow$ $H_n(M_f, X) \cong 0$ $\forall n$ by previous corollary

$\Rightarrow$ The maps $H_n(X) \xrightarrow{i_*} H_n(M_f)$ are all $\cong$ by LES on relative homology

$\Rightarrow$ $\begin{tikzcd} H_n(X) \arrow[r, "i_*"] \arrow[dr, "f_*"'] & H_n(M_f) \arrow[d, "\wr"] \\ & H_n(Y) \end{tikzcd}$, shows f_* is $\cong$ $\forall\, n$. //

Whitehead's Theorem

I made a claim to you that π_n was like the mother of all invariants in homotopy theory. Why did I say that?

<u>Thm</u> (Whitehead) Let X and Y be connected CW complexes. If a continuous map

$$f : X \to Y \quad , \quad f(x_0) = y_0$$

induces isomorphisms

$$f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0)$$

$\forall\, n$, then f is a homotopy equivalence.

We already saw this means f_* is an $\cong$ on homology. So it's natural (since all our known invariants are preserved) to ask for an inverse up to homotopy.

<u>Cor</u> If f induces an $\cong$ on π_n $\forall n$, then f also induces an $\cong$ on H_n $\forall n$.

(Since homotopy equivalences induce $\cong$ on $H_\bullet$.)

<u>Cor</u> Let X, Y be homotopy equivalent to CW complexes. If a map $f : X \to Y$ induces an $\cong$ on homotopy groups, it is a homotopy equivalence.

$$\left(\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "\wr", "g_Y"'] \\ X' \arrow[u, "\wr"', "g_X^{-1}"] & Y' \end{tikzcd}\right.$$

Then $g_Y \circ f \circ g_X^{-1}$ is a homotopy equivalence by Thm, so f is a htpy equivalence.)

<u>Cor</u> If X, Y are homotopy equivalent to CW complexes, X and Y are htpy equiv iff $\exists$ map $f : X \to Y$ inducing $\cong$ on $\pi_\bullet$.

<u>**Defn**</u> Let X and Y be CW complexes. A map $f: X \to Y$ is called <u>cellular</u> if

$$f(X^n) \subset Y^n \quad \forall\, n \geq 0.$$

<u>**Homework**</u> Every continuous map between CW complexes is homotopic to a cellular map.

<u>**Homework**</u> If $f: X \to Y$ is a cellular map, then M_f is a CW complex, and $X \subset M_f$ is a sub-CW-complex.

Pf (of Whitehead's theorem)

First assume $f: X \to Y$ is an inclusion.

$$\left[\begin{array}{l}\text{Apply compression lemma to} \\ \qquad (Y,X) \xrightarrow{id} (Y,X)\end{array}\right]$$ — IDEA

Since $f_*: \pi_n X \to \pi_n Y$ is an $\cong$ $\forall n$, ~~by LES~~

$$\pi_n(Y,X) \cong 0$$

by LES of relative π_n.

By compression lemma, id is homotopic rel X to a map factoring through X.

$\Rightarrow$ X is a deformation retract of Y

$\Rightarrow$ $f: X \to Y$ is a homotopy equivalence.

<u>Pf</u> (of Whitehead's Thm) continued:

By homework, f is homotopic to a cellular map

$$f' : X \to Y.$$

Then $(M_{f'}, X)$ is a CW pair.

By remarks,

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "f'", "\cong \text{ on } \pi_*"'] & M_{f'} \arrow[d, "\cong \text{ on } \pi_* \quad (\text{since def. retraction})"] \\
& Y
\end{tikzcd}$$

so $\quad X \hookrightarrow M_{f'}$ induces $\cong$ on π_n $\forall n$.

By lemma, $X \hookrightarrow M_{f'}$ is a homotopy equivalence,

hence so is $f' : X \to Y$.

Since $f' \sim f$, we're done. //

~~Next class, we'll prove Lemma One, which implies Lemma Two.~~

~~(~~

~~Compression Lemma~~

~~Interesting in its~~

~~own right.~~

Any homotopy equivalence $f: X \to Y$ induces

$\cong$ on π_n.

What if a map just induces $\cong$ on π_n?

Is it a homotopy equivalence in general?

No. So we give this a name.

Defn A continuous map $f: X \to Y$

is called a weak homotopy equivalence

if $\forall\, x_0 \in X$,

$$f_*: \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$$

is an $\cong$ $\forall\, n \geq 0$.

Rmk $n=0$ gives $\cong$ of set of connected components.

(CW approximation)

<u>Thm</u> Any space receives a weak homotopy equivalence from a CW complex.

i.e., $\forall$ Y, $\exists$ CW complex X and a map $f: X \longrightarrow Y$ s.t.

$$f_*: \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$$

is $\cong$ $\forall$ n, x_0.

<u>Pf</u> Assume Y is path connected (Otherwise, do this proof on each path-connected component.) Fix $X^0 = \{x_0\}$, and let $g_0(x_0) = y_0$.

For $n=1$, let $B_1 = \{\beta\}$ be a set of generators for $\pi_1(Y, y_0)$.

Let $\Sigma_1 := \bigvee_{B_1} S^1_\beta$.

This defines a map

$$g_1 : \Sigma_1 \longrightarrow Y$$
$$S^1_\beta \xrightarrow{\beta} Y$$

which is a surjection on π_1.

Let $C_1 = \{\gamma\}$ be a set of representatives of generators for the kernel

$$\text{Kernel}\,(\pi_1 Z_1 \longrightarrow \pi_1 Y).$$

Then define the space

$$X_1 = Z_1 \amalg \Big(\coprod_{C_1} D^2_\gamma \Big) \Big/ \sim$$

where $\sim$ is the relation gluing ∂D^2_γ to Z_1 via the map $\gamma : S^1 \cong \partial D^2_\gamma \longrightarrow Z_1$.

~~Then $f_1 : X_1 \longrightarrow Y_1$ is an $\cong$ on π_1.~~

Now, if γ is in the kernel, $\exists$ some map of disks

$$f_\gamma : (D^2, \partial D^2) \to Y$$

taking ∂D^2 to $g_1(\gamma)$.

The collection of f_γ defines a map

$$Z_1 \amalg \Big(\coprod_{C_1} D^2_\gamma \Big) \longrightarrow Y$$

which factors through X_1 by construction!

Call the factoring map

$$f_1 : X_1 \longrightarrow Y.$$

It's an $\cong$ on π_1.

By induction, assume we have a map

$$f_{k-1}: X_{k-1} \longrightarrow Y$$

which is an $\cong$ on $\pi_{k'}$, $\forall\ k' \leq k-1$.

Also assume

$$f_{k-1}\big|_{X_{k'}} = f_{k'} : X_{k'} \to Y$$

$\forall\ k' \leq k-1$.

Choose generators

$$B_{k} = \{\beta : S^{k} \to Y\}$$

of $\pi_k(Y)$, and let

$$Z_k := (X_{k-1}) \vee \left(\bigvee_{B_k} S^k_\beta \right).$$

The map

$$g_k : Z_k \longrightarrow Y$$

is a surjection on π_k.

(This map is defined so

$$g_k\big|_{X_{k-1}} = f_{k-1}$$

and

$$g_k\big|_{S^k_\beta} = \beta_k.)$$

Now let $C_k = \{\gamma\} = \{$Representative for each element/generator of $\ker(g_k : \pi_k(X) \to \pi_k(Y))\}$

This defines a map

$$Z_k \amalg \left(\coprod_{C_k} D^k_\gamma \right) \longrightarrow Y$$

which factors though

$$X_k := Z_k \amalg \left(\coprod_{C_k} D^k_\gamma \right) \Big/ \sim .$$

Set

$$X := \bigcup_{k \geq 0} X_k ;$$

w/ topology so $U \subset X$ open $\iff$ $U \cap X_k$ open.

$$f : X \longrightarrow Y$$
$$x_k \longmapsto f(x_k)$$
$$\rotatebox{90}{$\in$}$$
$$X_k$$

//

Rmk. $X_k \hookrightarrow X$

is an $\cong$ on $\pi_{k'}$

for $k' \leq k-1$.

This is by cellular approximation (your homework).

Giving $S^{k'}$ a CW structure w/ only cells in dimension $\leq k'$, cellular approx. tells us any

$$S^{k'} \longrightarrow X$$

is homotopic to a map factoring through $X_{k'}$.

Likewise, any "killing" disk

$$(D^{k'+1}, \partial D^{k'+1}) \longrightarrow X$$

factors through

$$(X^{k'+1}, X^{k'}).$$

Rmk Let Y be any space, and $A \subset Y$ a subspace given a CW structure.

Setting $X_0 = A$ in the proof, we obtain a map ~~of CW pairs~~

$$f: (X, A) \to (Y, A)$$

from a CW pair (X,A) to the pair (Y,A), and this induces an $\cong$ on $\pi_* A$, $\pi_* X \cong \pi_* Y$.

Cor If $\pi_{n'}(Y,A) = 0$ $\forall$ $n' \leq n$, and Y is connected, ~~then~~ and A is CW,

then $\exists$ CW pair (X,A) w/ $X \setminus A$ having cells only in dimension $\geq n+1$, s.t. $\exists$

$$f: (X,A) \xrightarrow{\ } (Y,A)$$

inducing $\cong$ on all homotopy groups.

Pf. $\pi_{n'}(Y,A) = 0$ $\forall n' \leq n$ $\Rightarrow$ $\pi_{n'}(A) \xrightarrow{\cong} \pi_{n'}(Y)$ $\forall n' < n$.

At $n' = n$, $\pi_n(A) \to \pi_n(Y) \to 0$ so $\pi_n(A) \to \pi_n(Y)$ is a surjection. So we can begin the induction at Z_{n+1}. //

[Excision for homotopy groups].

Defn. We say a pair of spaces (X,A) is <u>n-connected</u> if

- each path-connected component of X intersects A, and
- $\pi_i(X,A,x_0)=0 \quad \forall\ 1\le i\le n$.

Rmk.

$$\pi_n(A)\to\pi_n(X)\to\pi_n(X,A)\to$$
$$\to\pi_{n-1}(A)\to\pi_{n-1}(X)\to\pi_{n-1}(X,A)\to$$
$$\to\cdots$$
$$\pi_1(X,A)\to$$
$$\to\pi_0A\to\pi_0X$$

$\pi_nA\to\pi_nX$ is a surjection.

$\pi_iA\to\pi_iX$ is $\cong$ $\forall$ $1\le i\le n-1$.

surjection.

Thm Let X be CW.

Let $A,B\subset X$ be sub-CW complexes

s.t.

- $X=A\cup B$
- $C=A\cap B\neq\emptyset$.

[Excision]

If (A,C) is m_A-connected

(B,C) is m_B-connected

then $\pi_i(A,C)\to\pi_i(X,B)$ is $\cong$ $\forall$ $i<m+n$

surjection for $i=m+n$.

Rmk We proceed w/out a proof for now. I haven't decided whether to prove it in homework or in class.

There's another fundamental result, called the long exact sequence of a fibration.

So let's define a fibration.

Defn: A map

$$p : E \to B$$

is called a fibration if $\forall$ commutative diagrams

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "{(x,0)}"'] & E \arrow[d, "p"] \\ Z \times [0,1] \arrow[r] & B \end{tikzcd}$$

$\exists$ a dotted arrow

$$\begin{tikzcd} Z \arrow[r, "1"'] \arrow[d] & E \arrow[d] \\ Z \times [0,1] \arrow[r] \arrow[ur, dashed, "2"'] & B \end{tikzcd}$$

making 1 and 2 commutative

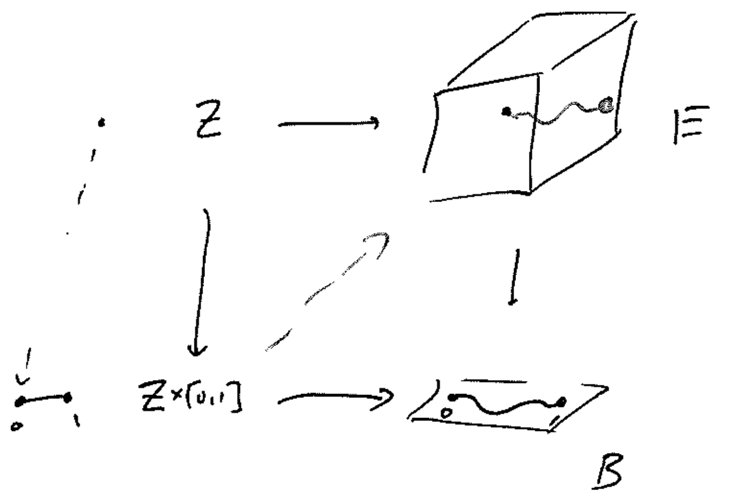
This should vaguely remind you of HEP.

$$\begin{tikzcd} Y & X \arrow[l] \arrow[dl, dashed] \\ Maps(I,Y) \arrow[u] & A \arrow[l] \arrow[u, hook] \end{tikzcd}$$

We've reversed all the arrows.

Before, we wiggled A inside of Y, and extended the wiggle to all of X.

Now, we wiggle some Z inside B, and we want to lift the wiggle to take place in E, casting a shadow down to B as the original wiggle.



Defn A map

$$p: E \to B$$

is called a *Serre fibration* if it satisfies the lifting property whenever Z is homeomorphic to a disk.

Rmk So being a Serre fibration is easier than being a fibration.

Prop'n Let B be path-connected, and let

$$p: E \to B$$

be a Serre fibration.

Fix

- $b_0 \in B$
- $F := p^{-1}(b_0)$
- $x_0 \in F$.

The map

$$\pi_i(E, F, x_0) \xrightarrow{\pi_i} (B, b_0, b_0) \cong \pi_i(B, b_0)$$

is an $\cong$ $\forall\, i$.

Pf.

Surjection.

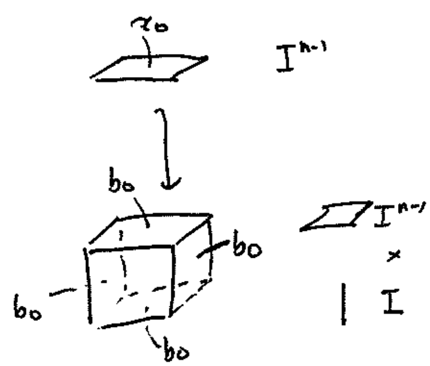
Consider the map

$$I^{n-1} \cong D^{n-1} \xrightarrow{x_0} E$$

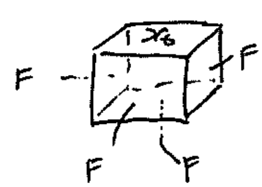
sending everything to x_0.

For any $f: (I^n, \partial I^{n-1}) \to (B, b_0)$ we have a commutative diagram

$$\begin{tikzcd} I^{n-1} \arrow[r, "x_0"] \arrow[d] & E \arrow[d] \\ I^{n-1} \times I \arrow[r, "f"] & B \end{tikzcd}.$$



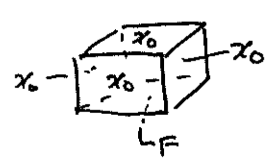
By lifting property, we have a map



,

$$\begin{tikzcd} & E \arrow[d] \\ I^{n-1} \times I \arrow[ur] \arrow[r, "f"] & B \end{tikzcd}$$

which we can homotope to a map

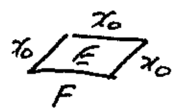


$g: (I^n, \partial I^n, J) \to (E, F, x_0)$

by "drooping" x_0 down.

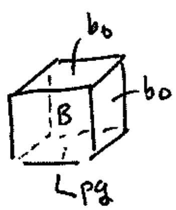
This homotopy gives a homotopy from f to f' rel b_0 in B, so $[g] \in \pi_n(E,F,x_0)$ is mapped to $[f] \in \pi_n(B,b_0)$.

<u>Injection</u>. Suppose



$g : (I^n, \partial I^n, J) \to (E, F, x_0)$

and that pg is homotopic to a constant map rel b_0.

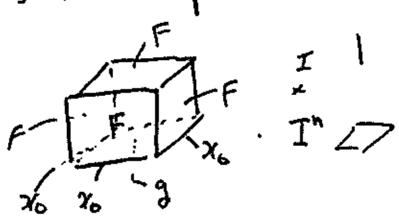


$h : (I^{n+1}, \partial I^n \times I) \to (B, b_0)$

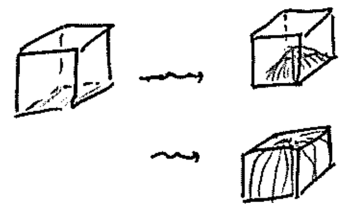
By lifting property,

$$\begin{tikzcd} I^n \arrow[r, "g"] \arrow[d] & E \arrow[d] \\ I^n \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

h lifts to a map



The retraction (deformation retraction) from $I^n \times I$ to "every face but the bottom face"



exhibits a homotopy from g to a map factoring through F.

Hence $[g] = 0 \in \pi_n(E,F,x_0)$. //

Now consider the LES

$$\to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(E, F, x_0) \to \pi_{n-1}(F) \to \cdots.$$

<u>**Cor**</u> (LES of a fibration).

Let B be path-connected, and $p : E \to B$ be a Serre fibration.

Then $\exists$ a long exact sequence

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots.$$

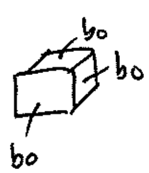
What are the morphisms?

- $\pi_n(E, x_0) \to \pi_n(E, F, x_0) \to \pi_n(B, b_0)$

  is just the group homomorphism associated to projection

  $p : E \to B$.

- $\pi_n(B, b_0) \to \pi_{n-1}(F, x_0)$

  Given 

$I^n \to B$

  lift to a map 

$: I^n \to E$

  and restrict to ∂I^n.

<u>Next</u>: Any fiber bundle is a fibration.