[Excision for homotopy groups].

Defn. We say a pair of spaces (X, A) is <u>n-connected</u> if

- each path-connected component of X intersects A, and
- $\pi_i(X, A, x_0) = 0 \quad \forall\ 1 \le i \le n$.

Rmk.

$$\pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X,A) \to$$
$$\to \pi_{n-1}(A) \to \pi_{n-1}(X) \to \pi_{n-1}(X,A) \to$$
$$\to \cdots$$
$$\pi_1(X,A) \to$$
$$\to \pi_0 A \to \pi_0 X$$

$\pi_n A \to \pi_n X$ is a surjection.

$\pi_i A \to \pi_i X$ is $\cong \ \forall\ 1 \le i \le n-1$.

surjection.

Thm Let X be CW.

Let $A, B \subset X$ be sub-CW complexes

s.t.

- $X = A \cup B$
- $C = A \cap B \neq \emptyset$.

[Excision]

If (A, C) is m-connected

(B, C) is n-connected

then $\pi_i(A, C) \longrightarrow \pi_i(X, B)$ is $\cong \ \forall\ i < m+n$

surjection for $i = m+n$.

__Rmk__ We proceed w/out a proof for now. I haven't decided whether to prove it in homework or in class.

There's another fundamental result, called the long exact sequence of a <u>fibration</u>.

So let's define a fibration.

__Defn__: A map

$$p : E \to B$$

is called a <u>fibration</u> if $\forall$ commutative diagrams

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "{(x,0)}"'] & E \arrow[d, "p"] \\ Z \times [0,1] \arrow[r] & B \end{tikzcd}$$

$\exists$ a dotted arrow

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "①"] & E \arrow[d] \\ Z \times [0,1] \arrow[r] \arrow[ur, dashed] \arrow[ur, phantom, "②", shift right=3] & B \end{tikzcd}$$

making ① and ② commutative

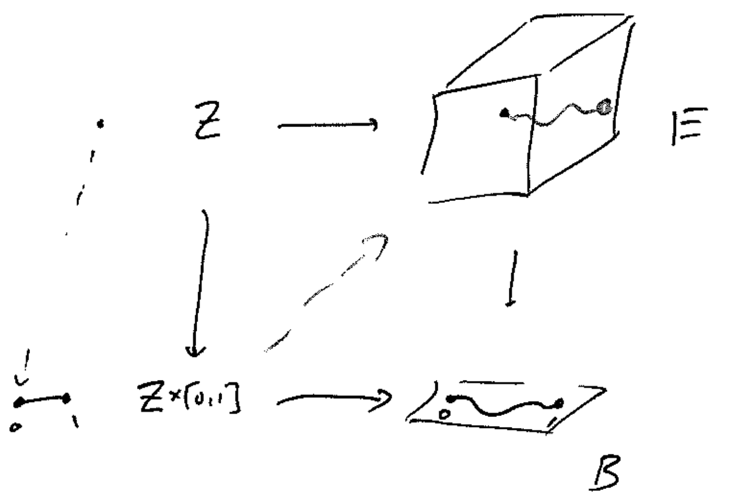
This should vaguely remind you of HEP.

$$\begin{tikzcd} Y & X \arrow[l] \arrow[dl, dashed] \\ \mathrm{Maps}(I,Y) \arrow[u] & A \arrow[l] \arrow[u, hook] \end{tikzcd}$$

We've <u>reversed</u> all the arrows.

Before, we wiggled A inside of Y, and <u>extended</u> the wiggle to all of X.

Now, we wiggle some Z inside B, and we want to <u>lift</u> the wiggle to take place in E, casting a shadow down to B as the original wiggle.



Defn A map

$$p : E \to B$$

is called a *Serre fibration* if it satisfies the lifting property whenever Z is homeomorphic to a disk.

Rmk So being a Serre fibration is easier than being a fibration.

Prop'n Let B be path-connected, and let

$$p : E \to B$$

be a Serre fibration.

Fix

- $b_0 \in B$
- $F := p^{-1}(b_0)$
- $x_0 \in F$.

The map

$$\pi_i(E, F, x_0) \xrightarrow[\pi_i]{} (B, b_0, b_0) \cong \pi_i(B, b_0)$$

is an $\cong$ $\forall i$.

Pf.

Surjection.

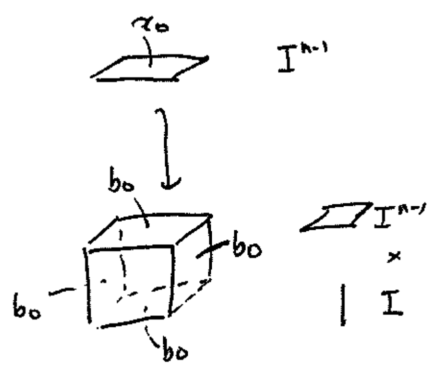
Consider the map

$$I^{n-1} \cong D^{n-1} \xrightarrow{x_0} E$$

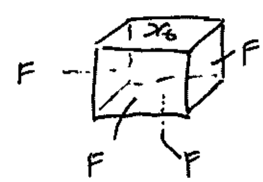
sending everything to x_0.

For any $f : (I^n, \partial I^{n-1}) \to (B, b_0)$ we have a commutative diagram

$$\begin{tikzcd} I^{n-1} \arrow[r, "x_0"] \arrow[d] & E \arrow[d] \\ I^{n-1} \times I \arrow[r, "f"] & B \end{tikzcd}$$



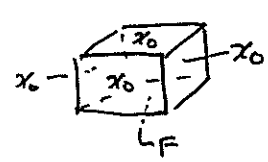
By lifting property, we have a map



,

$$\begin{tikzcd} & E \arrow[d] \\ I^{n-1} \times I \arrow[ur] \arrow[r, "f"'] & B \end{tikzcd}$$

which we can homotope to a map

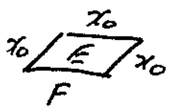


$g : (I^n, \partial I^n, J) \to (E, F, x_0)$

by "drooping" x_0 down.

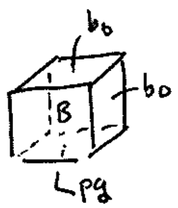
This homotopy gives
a homotopy from f to f' rel b_0
in B, so $[g] \in \pi_n(E, F, x_0)$
is mapped to $[f] \in \pi_n(B, b_0)$.

Injection. Suppose



$g: (I^n, \partial I^n, J) \to (E, F, x_0)$

and that pg is homotopic to
a constant map rel b_0.

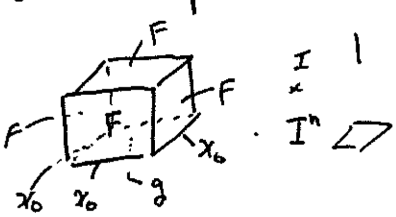


$h: (I^{n+1}, \partial I^n \times I) \to (B, b_0)$.

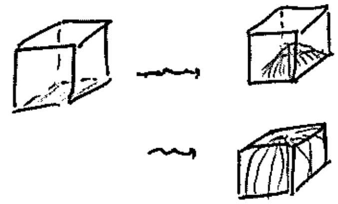
By lifting property,

$$\begin{tikzcd} I^n \arrow[r, "g"] \arrow[d] & E \arrow[d] \\ I^n \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

h lifts to a map



The retraction (deformation retraction)
from $I^n \times I$ to "every face
but the bottom face"



exhibits a homotopy from g
to a map factoring through F.

Hence $[g] = 0 \in \pi_n(E, F, x_0)$. //

Now consider the LES

$$\to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(E, F, x_0) \to \pi_{n-1}(F) \to \cdots.$$

Cor (LES of a fibration).

Let B be path-connected,
and $p: E \to B$ be a Serre fibration.
Then $\exists$ a long exact sequence

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots.$$

What are the morphisms?

- $\pi_n(E, x_0) \longrightarrow \pi_n(E, F, x_0) \to \pi_n(B, b_0)$

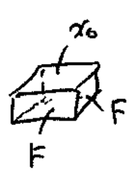
  is just the group homomorphism associated to projection

  $$p: E \to B.$$

- $\pi_n(B, b_0) \longrightarrow \pi_{n-1}(F, x_0)$

  Given 

 $I^n \to B$

  lift to a map 

 $: I^n \to E$

  and restrict to ∂I^n.

Next: Any fiber bundle is a fibration.