Serre fibrations
and
fiber bundles

Recall from last time:

<u>Defn</u> A map $p: E \to B$ is called a Serre fibration if $\forall$ diagrams

$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

the dotted arrow always exists.

<u>Exer</u> If B is path-connected, and E is non-empty,
show a Serre fibration is a surjection.

Prop.

If $p: E \to B$ is a Serre fibration, for any diagram

$$\begin{tikzcd} D^n \times \{0\} \cup \partial D^n \times I \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

the dashed arrow lift exists.

Pf. $D^n \times \{0\} \cup \partial D^n \times I$ is homeomorphic to D^n.

 $\cong$ 

$\cong$  D^n. //

Cor For any Serre fibration,

and any diagram

$$\begin{tikzcd}
\partial D^n \arrow[r, hook] \arrow[d] & D^n \arrow[r, "h"] \arrow[d] & E \arrow[d, "p"] \\
\partial D^n \times I \arrow[r, hook] \arrow[urr, dashed, "\alpha"] & D^n \times I \arrow[r] \arrow[ur, dashed, "\beta"'] & B
\end{tikzcd}$$

$\exists \alpha \implies \exists \beta$.

Pf To give maps α, h in diagram above

is the same thing as

giving a map

$$\partial D^n \times I \cup D^n \times \{0\} \longrightarrow E$$

as in proposition. //

Cor For any Serre fibration, and any CW pair (X, A)

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ A\times I \arrow[r, hook] \arrow[urr, dashed, "\alpha"] & X\times I \arrow[r] \arrow[ur, dashed, "\beta"'] & B \end{tikzcd}$$

$\exists \alpha \Rightarrow \exists \beta.$

Pf. Construct β on

$X^n \cap (X \setminus A)$.

Then extend to

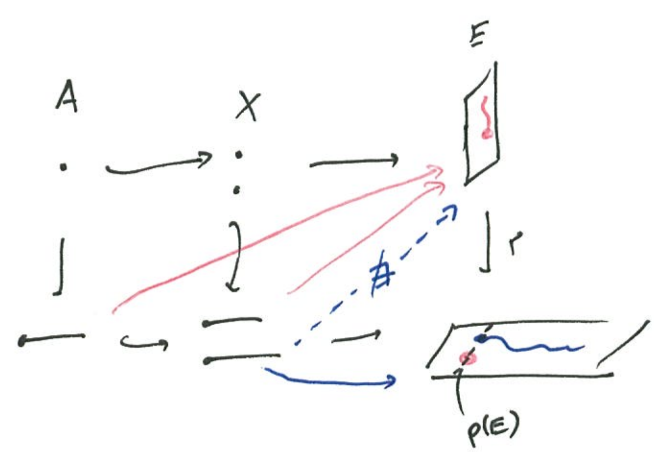
$X^{n+1} \cap (X \setminus A)$

using previous corollary.

Proceed by induction.

Rmk If $B = *$, we have the homotopy extension property of (X, A). But HEP alone cannot guarantee these lifts; p must be a fibration!

Ex



Defn A continuous map

$$p: E \to B$$

is called a <u>fiber bundle</u> if ∃ a space F s.t. ∀ $b \in B$, ∃ an ~~homeomorphism~~ open set $U_b \subset B$ and a homeomorphism

$$E \supset p^{-1}(U_b) \cong F \times U_b$$

such that

$$\begin{tikzcd}
F \times U_b \arrow[r, phantom, "\cong"] \arrow[dr, "\pi"'] & p^{-1}(U_b) \subset E \arrow[d, "p"] \\
& U_b
\end{tikzcd}$$

commutes.

U_b is called a <u>trivializing</u> neighborhood.

Ex. If $E \cong F \times B$,

then

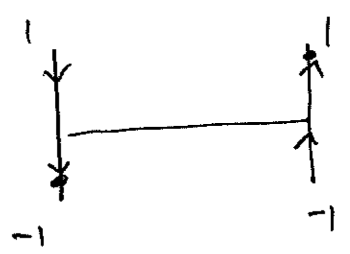
$$p : E \longrightarrow B$$
$$(x, b) \longmapsto b$$

is a fiber bundle, called the <u>trivial fiber bundle</u>.

Exer Show Mobius band is homotopy equivalent to the cylinder.

Ex Consider the Mobius band

$$E = [0,1] \times [-1,1] \big/ (0,t) \sim (1,-t).$$



$\cong$

one twist.

$\exists$ map $E \longrightarrow S^1 \cong B$
$(x, t) \longmapsto x$

w/ $F \cong [-1,1]$. But $\partial E \cong S^1$, while $\partial(S^1 \times [-1,1]) \cong S^1 \sqcup S^1$, so this is <u>NOT</u> the trivial bundle.

Ex A covering space

$$p: E \to B$$

is a fiber bundle, where fibers are discrete.

Ex. The Hopf fibration

$$S^3 \longrightarrow S^2$$

is a fiber bundle w/ fiber S^1. (Recall this popped up when we defined $\mathbb{C}P^2$.)

In general,

$$S^{2n+1} \longrightarrow \mathbb{C}P^n$$

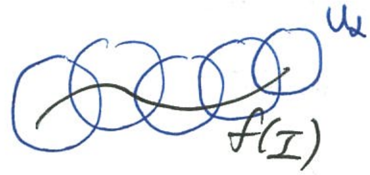
is a fiber bundle w/ fiber S^1.

Prop'n Every fiber bundle is a Serre fibration.

Pf By induction on the n in $D^n \cong I^n$.

For $n=0$:

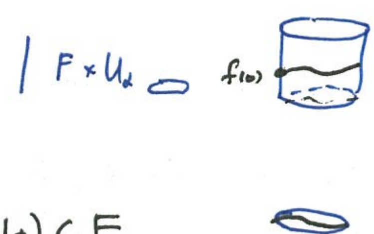
$$\begin{tikzcd} D^0 \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ D^0 \times I \arrow[r, "f"] & B \end{tikzcd}$$



Fix an open cover of $f(I)$ by trivializing n-hoods U_α.

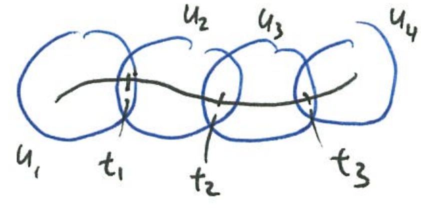
Since I compact, we choose finite subcover of $\{f^{-1}(U_\alpha)\}$.

For the interval $[0, t_1)$ w/ image in a trivializing n-hood, define



$$t \xmapsto{\tilde{f}} (f(0), t) \in F \times U_1 \xrightarrow[\sim]{\Phi_1^{-1}} p^{-1}(U_1) \subset E.$$

Assume we've chosen $t_i \in [0,1]$ so $[t_i, t_{i+1}]$ lands in a single trivializing n-hood $\forall\, i$.



$$\begin{tikzcd} p^{-1}(U_\alpha \cap U_\beta) \arrow[r, "\cong"] \arrow[d, "\cong" labelled, "\varphi_i"'] & F\times U_{i-1} \\ F\times U_i & \end{tikzcd}$$

$$p^{-1}(U_\alpha\cap U_\beta) \overset{\varphi_{i-1}}{\cong} F\times U_{i-1}$$

$$p^{-1}(U_\alpha\cap U_\beta) \overset{\varphi_{i}}{\cong} F\times U_{i}$$

By induction, define

$$\tilde{f}_i : [t_{i-1}, t_i] \longrightarrow E$$

by

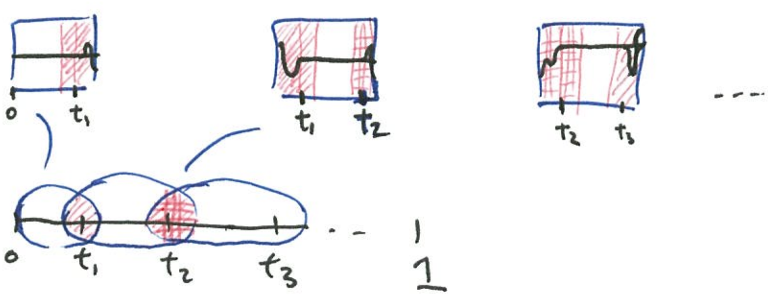
$$\tilde{f}_i(t) = \varphi_i^{-1}\overbrace{\Big(\pi_F \varphi_i\big(\underset{\substack{\cap \\ p^{-1}(U_{i-1}\cap U_i)}}{\tilde{f}_{i-1}(t_{i-1})}\big)\ ,\ f(t)\Big)}^{F\times U_i \ \ni}.$$

Fancy way of writing,

"Constant path in F component", "do f in U component"

only looks constant in the trivialization φ_i.

i.e., B component.

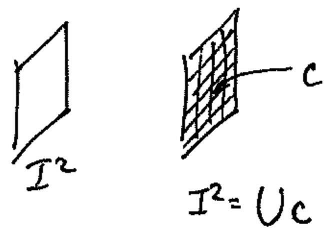


Now given

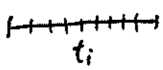
$$\begin{tikzcd} I^n \arrow[r, "\tilde{f}_0"] \arrow[d] & E \arrow[d] \\ I^n \times I \arrow[r, "f"] & B \end{tikzcd}$$

$\exists$ a subdivision of I^n

into cubes

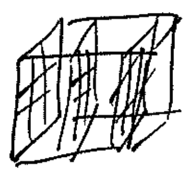


and of I into intervals



s.t.

$$f(C \times [t_i, t_{i+1}])$$



lands in a trivializing n-hood.

By induction on n, assume we've

defined a lift for

$$\partial C \times [0, t_1], \quad \tilde{g} : \partial C \times [0,t_1] \longrightarrow E$$

$$\begin{tikzcd} \partial C \times [0,t_1] \arrow[r] \arrow[d] & E \\ U_\alpha \times F \arrow[r, "\cong", "\phi_\alpha"'] & p^{-1}(U_\alpha) \arrow[u] \end{tikzcd}$$

To define $\tilde{f}$ on $C \times [0, t_1]$,

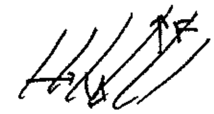
take the map

$$\tau : C \times [0,t_1] \xrightarrow{\text{retract}} C \times \{0\} \cup \partial C \times [0,t_1] \xrightarrow{\tilde{f}_0 \cup \tilde{g}} p^{-1}(U_\alpha) \xrightarrow{\sim} F \times U_\alpha \longrightarrow F.$$

The map

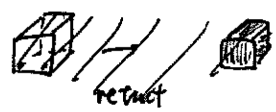
$$C\times[0,t_1] \xrightarrow{\tau\times f} F\times U_\alpha \cong p^{-1}(U_\alpha)$$

defines $\tilde f|_{C\times[0,t_1]}$.



Proceed by induction on t_i to define

$\tilde f$ on $C\times[t_i,t_{i+1}]$. //



Ex

Let $\mathbb{C}^\infty$ be the set of all

$$(z_0, z_1, z_2, \dots) \in Maps(\mathbb{Z}_{\geq 0}, \mathbb{C})$$

s.t. $z_i = 0$ for all but finitely many i.

Topologize as

$$\mathbb{C}^\infty = \bigcup_{i \geq 1} \mathbb{C}^i, \qquad U \text{ open} \Leftrightarrow U \cap \mathbb{C}^i \text{ open } \forall i$$

Let $S^\infty = \{\vec{z} \in \mathbb{C}^\infty \mid \|z\|^2 = 1\}$.

Mod out by equiv relation

$$\vec{z} \sim e^{i\theta}\vec{z} \qquad , e^{i\theta} \in S^1.$$

$$(z_0, z_1, \dots) \sim (e^{i\theta}z_0, e^{i\theta}z_1, \dots)$$

Call resulting space $\mathbb{CP}^\infty$.

By defn,

$$\mathbb{CP}^\infty = \bigcup_{i \geq 1} \mathbb{CP}^{i-1}, \qquad U \text{ open} \Leftrightarrow U \cap \mathbb{CP}^i \text{ open } \forall i.$$

This gives a fiber bundle

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^\infty \\ & & \downarrow \\ & & \mathbb{CP}^\infty. \end{array}$$

Note by def'n,

$$S^\infty = \bigcup_{i\geq 1} S^{2i-1}$$

$$\cong \bigcup_{j\geq 0} S^j$$

obviously homeomorphic.

So S^∞ is a CW complex.

(You can give S^n a CW structure s.t.
$S^{n-1} \hookrightarrow S^n$ is a CW subcomplex.)

<u>Exer</u>

(1) Show $\pi_n(S^\infty) = 0 \ \forall\, n \geq 0$

(2) Show S^∞ is homotopy equivalent to a point.

(3) Compute $\pi_n(\mathbb{C}P^\infty) \ \forall\, n \geq 0$.

(4) Show S^2 and $S^3 \times \mathbb{C}P^\infty$ have $\cong$ homotopy groups in all dimensions.

(Homework: $\pi_*(X \times Y) \cong \pi_* X \times \pi_* Y$.)

(5) Show S^2 is not homotopy equivalent to $S^3 \times \mathbb{C}P^\infty$.

Solutions

(1) Given $f: S^k \to S^\infty$,

by cellular approximation, f is homotopic to f' s.t. $f'(S^k) \subset S^k$.

Alternatively, by compactness of S^k, $f(S^k) \subset S^N$ for some large N.

But then $f': S^k \to S^k \hookrightarrow S^{k+1} \subset S^\infty$

and $\pi_k S^{k+1} = 0$ by cellular approximation.

So any map f is homotopic to something that factors through the zero map on π_k.

(2) $*$ and S^∞ are both CW complexes. By Whitehead, $* \hookrightarrow S^\infty$ is hence a homotopy equivalence.

(3) $F = S^1$

$E = S^\infty \simeq *$

$B = \mathbb{CP}^\infty$.

$$\pi_n(S^1) \to \pi_n(S^\infty) \to \pi_n(\mathbb{CP}^\infty) \to \pi_{n-1}(S^1)$$

If $n-1 \geq 2$, all groups in sight are $\cong$ to 0.

If $n-1 = 1$, we see
$$\pi_2(\mathbb{CP}^\infty) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

And $\pi_1 \mathbb{CP}^\infty \cong 0$, by cellular approx or above LES.

$$\pi_n \mathbb{CP}^\infty = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

(4) $S^1 \hookrightarrow S^3 \to S^2$ $\Rightarrow$ $\pi_n(S^3) \cong \pi_n(S^2)$ $\forall\, n \geq 3$,

$\pi_2(S^3 \times \mathbb{CP}^\infty) \cong \pi_2 S^3 \times \pi_2 \mathbb{CP}^\infty \cong 0 \times \mathbb{Z} \cong \pi_2 S^2$

$\pi_1(\quad) \cong \pi_1 \times \pi_1 \cong 0 \cong \pi_1 S^2$. //

(5) S^3 has no higher homology. $\mathbb{CP}^\infty$ does OR: $H_3(S^3) \neq 0$.