Compactly Generated Spaces

Basic Defns:

<u>Prop</u> Let K be compact,
$f: K \to X$ continuous.
If X is Hausdorff, $f(K) \subset X$
is closed.

Motivated by this,

<u>Defn</u> X is <u>weak</u> <u>Hausdorff</u>
if $\forall$ K compact Hausdorff,
$f: K \to X$ continuous,
$f(K) \subset X$ is closed.

<u>Ex</u>. If X is weak Hausdorff,
any point is a closed subset.

(So $\mathrm{Spec}\, R$ is NOT weak Hausdorff.)

Defn $A \subset X$ is <u>compactly</u> <u>closed</u> (or k-closed)
if $\forall$ $g: K \to X$, K compact Hausdorff,
$g^{-1}(A) \subset K$ is closed.

Rmk Any closed $A \subset X$ is compactly closed.

Defn X is called a <u>k space</u> if
A closed $\Leftrightarrow$ A compactly closed.

<u>A non-example:</u> (Really, just non-Hausdorff.)
Consider $\mathbb{C}^\times \curvearrowright S^2 \cong \mathbb{C}P^1$ by

$$\alpha \cdot [z_0 : z_1] = [\alpha z_0 : z_1].$$

Thinking of S^2 as $\mathbb{C} \cup \{\infty\}$, this is usual action of $\mathbb{C}^\times$ on $\mathbb{C}$.
(Fixing $0 \in \mathbb{C}$ and $\infty \in S^2$.)

Then $\mathbb{C}P^1 / \mathbb{C}^\times$ is a space w/ three points: $\{0, \infty, \cdot\}$.

Open sets: $\{\cdot\}$, $\{\cdot, 0\}$, $\{\cdot, \infty\}$, $\{\cdot, 0, \infty\}$, ϕ

Closed sets: $\{0, \infty\}$, $\{\infty\}$, $\{0\}$, ϕ, $\{\cdot, 0, \infty\}$.

Rmk

$T^1 \Leftarrow$ weak Hausdorff $\Leftarrow$ Hausdorff

(points closed)

One rarely encounters a weak Hausdorff space that isn't closed.

Defn X is compactly generated if X is a weak Hausdorff k-space.

Ex Any compact Hausdorff space is compactly generated.

- Hausdorff $\Rightarrow$ weak Hausdorff
- Take $g = id : K \to K$
  $\Rightarrow$ every compactly closed set is closed.

Ex Any metric space. For simplicity, assume closed balls are compact. (Huge restriction.)

For more general metric spaces, deal w/ sequentially closed subsets.

- metric $\Rightarrow$ Hausdorff $\Rightarrow$ weak Hausdorff
- Set $g_n : \overline{B_n(0)} \hookrightarrow X$. ($\overline{B_n(0)}$ = closed ball of radius n.)

$A \cap \overline{B_n(0)}$ is closed $\forall n$ if A is compactly closed.

So A is closed

($\forall q \in X,\ \varepsilon > 0,\ B_\varepsilon(q) \subset \overline{B_{dist(0,q)+2\varepsilon}}(0)$

so complement of A is open.)

Prop'n Let X be compactly generated, Y arbitrary space.

Then $f: X \to Y$ is continuous iff $f|_K : K \to Y$ is continuous $\forall\, K \subset X$ compact, Hausdorff.

Rmk This explains "compactly generated." The topology of a space is determined by the continuous maps out of it, and the proposition shows that the compact subsets control the continuous maps out of X.

Pf. If f continuous, obviously $K \hookrightarrow X \to Y$ is continuous $\forall\, K \subset X$.

Conversely: Given

$$L \xrightarrow{j} X, \quad L \text{ compact Hausdorff},$$

the composite

$$\begin{tikzcd} L \arrow[r, "j"] \arrow[rr, bend right, "g_L"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

is continuous. $\left(\begin{tikzcd} L \arrow[r, "g_L"] \arrow[d] & X \\ \mathrm{image}(g) \arrow[ur, hook] & \end{tikzcd} \right.$, image$(g)$ compact, so $L \to \mathrm{image}(g) \to Y$ continuous.)

So $\forall\, B \subset Y$ closed, $g_L^{-1}(B) \subset L$ closed.

$\Rightarrow g_L^{-1}(B)$ has closed image in X by weak Hausdorff property.

$\Rightarrow f^{-1}(B)$ is compactly closed

$\Rightarrow f^{-1}(B)$ closed by k-space property. //

Recall: Given a collection of spaces $\{X_\alpha\}_{\alpha\in A}$ (A might even be uncountable) the space

$$\coprod_\alpha X_\alpha$$

is topologized so it's the finest topology where

$$X_\alpha \hookrightarrow \amalg X_\alpha$$

is continuous $\forall \alpha$.

Equivalently, $A \subset \amalg X_\alpha$ is closed/open iff $A \cap X_\alpha$ is closed/open $\forall\, \alpha$.

(In particular, $X_\alpha \subset \amalg X_\alpha$ is both open and closed.)

Exer (1) If X_α is a k-space $\forall \alpha$, so is $\amalg X_\alpha$.

(2) If X_α is weak Hausdorff $\forall \alpha$, so is $\amalg X_\alpha$.

Pf (1) If $A \subset \amalg X_\alpha$ satisfies $g^{-1}(A) \subset K$ closed $\forall\; g: K \to \coprod_\alpha X_\alpha$,

NTS: $A \cap X_\alpha$ closed $\forall\, \alpha$.

$\forall \alpha$, $K \xrightarrow[g']{} X_\alpha$ C^0 $\Rightarrow$ $\underbrace{K \to X_\alpha \hookrightarrow \amalg X_\alpha}_{g}$ C^0;

and $g^{-1}(A) = (g')^{-1}(A \cap X_\alpha)$ is closed by hyp.

$\Rightarrow A \cap X_\alpha$ is k-closed $\Rightarrow A \cap X_\alpha$ closed since X_α is a k-space.

(2) $\mathrm{image}(g) = \bigcup_{\substack{\text{finite}\\ \alpha}} \mathrm{image}(g) \cap X_\alpha$ = finite union of closed sets.

(i.e., is closed.) //

$= \mathrm{image}(g|_{g^{-1}(X_\alpha)})$

[illegible] cmpct Haus.

Lemma ① Let X_i be compactly generated, $i = 1, 2, 3, \cdots$

and let $f_i : X_i \to X_{i+1}$ be continuous maps s.t. f_i injections, and

$f_i(X_i) \subset X_{i+1}$ is closed. Then

$$X = \bigcup_i X_i \qquad \left(\text{w/ topology } \begin{array}{c} U \subset X \text{ open} \\ \Updownarrow \\ U \cap X_i \text{ open } \forall i \end{array} \right)$$

is compactly generated.

Lemma ② Let $A \subset X$ be closed.

If X and Y are compactly generated,

and $f : A \to Y$ is continuous, then

$$X \bigcup_f Y := X \amalg Y / a \sim f(a)$$

is compactly generated.

Cor CW complexes are compactly generated.

Pf Lemma ② says $X^{n-1} \bigcup_{\Phi} (\amalg D^n_\alpha) = X^n$

is compactly generated. (X^{n-1} by induction, and D^n_α is compactly generated since it's, for instance, compact Hausdorff.

And you can easily check $\amalg$ of arbitrarily many compactly generated spaces is again compactly gen.)

Lemma ① says $X = \bigcup X^n$ is.

//

__Defn__ Given a space X,
let kX be the topological space where

- $B \subset kX$ closed iff $B \subset X$ is compactly closed.

(kX has same underlying space as X.)

__Propn__ kX is a topological space.

__Pf.__ $g: L \to X$, L cmpct Hausdorff.

$$g^{-1}\left(\bigcap_{\alpha} A_\alpha\right) = \bigcap_{\alpha} g^{-1}(A_\alpha) \quad \text{closed.}$$

$$g^{-1}\left(\bigcup_{1,\dots,k} A_i\right) = \bigcup_{i=1,\dots,k} g^{-1}(A_i) \quad \text{closed.}$$

//

__Propn__ $A \subset kX$ is compactly closed $\Leftrightarrow$ $A \subset X$ is

__Pf__ Obviously $kX \to X$ is a continuous map, since $B \subset X$ closed $\Rightarrow$ $B \subset X$ compactly closed.
By defn of "compactly closed," any cont map

$$K \xrightarrow{g} X$$

factors through kX.

$$\begin{tikzcd} & kX \arrow[dr] & \\ K \arrow[ur, "g'"] \arrow[rr, "g"'] & & X \end{tikzcd}$$

(Assuming $A \subset kX$ cmpctly closed, any $g: K \to X$ has $g^{-1}(A) = (g')^{-1}(A) \subset K$ which is closed.)

So $A \subset kX$ compactly closed $\Rightarrow$ $A \subset X$ cmpctly closed. //

<u>Defn</u> Let

$$\mathrm{Spaces}^{wH}$$

be the category of weak Hausdorff spaces and cont. maps between them. Let

$$\mathrm{Spaces}^{cg}$$

denote the category of compactly generated spaces ~~maps~~ w/ morphisms continuous maps.

<u>Prop</u> k defines a functor

$$\mathrm{Spaces}^{cg} \longleftarrow \mathrm{Spaces}^{wH} : k$$

and

$$\mathrm{Maps}(X, kY) \cong \mathrm{Maps}(X, Y)$$

$$\forall \quad X \in \mathrm{Spaces}^{cg}, \quad Y \in \mathrm{Spaces}^{wH}.$$

Take this for granted for now, and let's define mapping spaces.

~~Defn Let X, Y be spaces.~~

~~Define $\mathrm{Maps}'(X,Y) = \{f: X \to Y,\ f \text{ cont.}\}$~~

to ~~have the compact open topology.~~

~~following topology : $\forall$ $K \subset X$ compact Hausdorff~~

~~$U \subset Y$ open,~~

~~let $C(K,U) = \{f \mid f(K) \subset U,\ f: X \to Y \text{ cont.}\}$.~~

<u>Defn</u> Let X, Y be topological spaces.

$\forall$ $K \subset X$ compact Hausdorff,

$\mathcal{U} \subset Y$ open, set

$$C(K, U) := \{ f : X \to Y \text{ contin.} \mid f(K) \subset U \}.$$

<u>Defn</u> Let $Maps'(X, Y)$ be the topological space for which

$$Maps'(X, Y) = \{ f : X \to Y \text{ contin.} \}$$

as a set, and the topology is generated by finite intersections of $C(K_i, U_i)$.

This topology is called the <u>compact-open topology</u>.

<u>Defn</u> Let

$$Maps(X, Y) := k\, Maps'(X, Y).$$

$$X \times Y := k(X \underset{\text{usual}}{\times} Y)$$

$$\underset{\text{subspace}}{A \subset X} := k(A \subset X).$$

With this notation,

Prop'n Let $X, Y, Z \in \text{Spaces}^{cg}$.

Then $\exists$ a homeomorphism

$$\text{Maps}(X \times Y, Z) \cong \text{Maps}(X, \text{Maps}(Y, Z)).$$

Reminder: This means

$$k\text{Maps}'(k(X \times^{\text{usual}} Y), Z) \cong k\text{Maps}'(X, k\text{Maps}'(Y, Z)).$$